Stochastic processes with imprecise probabilities: 
A case study involving Markov chains

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LMS Lecture
Newcastle University
29 May 2009
Precise probability models

Mass functions and expectations

Assume we are uncertain about:

- the value or a variable $X$
- in a set of possible values $\mathcal{X}$.

This is usually modelled by a **probability mass function** $p$ on $\mathcal{X}$:

$$p(x) \geq 0 \quad \text{and} \quad \sum_{x \in \mathcal{X}} p(x) = 1;$$

With $p$ we can associate an **expectation operator** $E_p$:

$$E_p(f) := \sum_{x \in \mathcal{X}} p(x) f(x) \quad \text{where} \quad f : \mathcal{X} \to \mathbb{R}.$$

If $A \subseteq \mathcal{X}$ is an **event**, then its **probability** is given by

$$P_p(A) = \sum_{x \in A} p(x) = E_p(I_A).$$
Precise probability models
The simplex of all probability mass functions

Consider the simplex $\Sigma X$ of all mass functions on $X$:

$$
\Sigma X := \left\{ p \in \mathbb{R}^X_+ : \sum_{x \in X} p(x) = 1 \right\}.
$$
Precise probability models
Geometrical interpretation of expectation

Assessments lead to constraints

Specifying an expectation $E(f)$ for a map $f : \mathcal{X} \to \mathbb{R}$

$$\sum_{x \in \mathcal{X}} p(x)f(x) = E(f)$$

imposes a linear constraint on the possible values for $p$ in $\Sigma \mathcal{X}$.

It corresponds to intersecting the simplex $\Sigma \mathcal{X}$ with a hyperplane, whose direction depends on $f$:

$$E(2I_{\{b\}} - I_{\{c\}}) = 0$$

$$E(I_{\{a\}}) = 1/2$$
Imprecise probability models

More flexible assessments

Impose linear inequality constraints on \( p \) in \( \Sigma X \):

\[
E(f) \leq \sum_{x \in X} p(x)f(x) \quad \text{or} \quad \sum_{x \in X} p(x)f(x) \leq E(f).
\]

Corresponds to intersecting \( \Sigma X \) with affine semi-spaces:
Imprecise probability models

Linear inequality constraints

More flexible assessments

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Imprecise probability models

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Imprecise probability models

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Impose **linear inequality constraints** on \( p \) in \( \Sigma \mathcal{X} \):

\[
\bar{E}(f) \leq \sum_{x \in \mathcal{X}} p(x)f(x) \quad \text{or} \quad \sum_{x \in \mathcal{X}} p(x)f(x) \leq \bar{E}(f).
\]

Corresponds to intersecting \( \Sigma \mathcal{X} \) with **affine semi-spaces**:
Imprecise probability models

More flexible assessments

Impose linear inequality constraints on $p$ in $\Sigma_X$:

\[
E(f) \leq \sum_{x \in X} p(x)f(x) \quad \text{or} \quad \sum_{x \in X} p(x)f(x) \leq E(f).
\]

Corresponds to intersecting $\Sigma_X$ with affine semi-spaces:
Any such number of assessments leads to a credal set $M$.

**Definition**

A credal set $M$ is a **convex closed** subset of $\Sigma_X$. 

![Diagram showing a convex closed subset](image)
Imprecise probability models

Lower and upper expectations

Consider the set $\mathcal{L}(\mathcal{X}) = \mathbb{R}^\mathcal{X}$ of all real-valued maps on $\mathcal{X}$. We define two real functionals on $\mathcal{L}(\mathcal{X})$: for all $f : \mathcal{X} \to \mathbb{R}$

\[
\begin{align*}
E_\mathcal{M}(f) &= \min \left\{ E_p(f) : p \in \mathcal{M} \right\} \quad \text{lower expectation} \\
\overline{E}_\mathcal{M}(f) &= \max \left\{ E_p(f) : p \in \mathcal{M} \right\} \quad \text{upper expectation}.
\end{align*}
\]

Observe that

\[
\overline{E}_\mathcal{M}(f) = -E_\mathcal{M}(-f).
\]
Imprecise probability models
Basic properties of upper expectations

**Definition**

We call a real functional $\overline{E}$ on $\mathcal{L}(\mathcal{X})$ an **upper expectation** if it satisfies the following properties:
for all $f$ and $g$ in $\mathcal{L}(\mathcal{X})$ and all real $\lambda \geq 0$:

1. $\overline{E}(f) \leq \max f$ [boundedness];
2. $\overline{E}(f + g) \leq \overline{E}(f) + \overline{E}(g)$ [sub-additivity];
3. $\overline{E}(\lambda f) = \lambda \overline{E}(f)$ [non-negative homogeneity].

**Theorem**

A real functional $\overline{E}$ is an upper expectation if and only if it is the upper envelope of some credal set $\mathcal{M}$.

**Proof.**

Use $\mathcal{M} = \{ p \in \Sigma_{\mathcal{X}} : (\forall f \in \mathcal{L}(\mathcal{X}))(E_p(f) \leq \overline{E}(f)) \}$. 

De Cooman (UGent)
We consider an uncertain process with variables $X_1, X_2, \ldots, X_n, \ldots$ assuming values in a finite set of states $\mathcal{X}$.

This leads to a standard event tree with nodes

$$s = (x_1, x_2, \ldots, x_n), \quad x_k \in \mathcal{X}, \quad n \geq 0.$$
Discrete-time uncertain processes

Precise probability trees

The standard event tree becomes a probability tree by attaching to each node a local probability mass function $p_s$ on $X$ with associated expectation operator $E_s$. 

De Cooman (UGent)

Imprecise Markov chains
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Discrete-time uncertain processes

Precise probability trees

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Precise probability trees
Calculating global expectations from local ones

Consider a function $g : \mathcal{X}^n \to \mathbb{R}$ of the first $n$ variables:

$$g = g(X_1, X_2, \ldots, X_n)$$

We want to calculate its expectation $E(g|s)$ in $s = (x_1, \ldots, x_k)$.

**Theorem (Law of Iterated Expectation)**

Suppose we know $E(g|s, x)$ for all $x \in \mathcal{X}$, then we can calculate $E(g|s)$ by backwards recursion using the local model $p_s$:

$$E(g|s) = E_s(E(g|s, \cdot)) = \sum_{x \in \mathcal{X}} p_s(x)E(g|s, x).$$

$$E(g|s) = p_s(a)E(g|s, a) + p_s(b)E(g|s, b)$$
Precise probability trees
Calculating global expectations from local ones

All expectations $E(g|x_1, \ldots, x_k)$ in the tree can be calculated from the local models as follows:

1. **Start in the final cut $\mathcal{X}^n$ and let:**

   $$E(g|x_1, x_2, \ldots, x_n) = g(x_1, x_2, \ldots, x_n);$$

2. **Do backwards recursion using the Law of Iterated Expectation:**

   $$E(g|x_1, \ldots, x_k) = E(x_1, \ldots, x_k) \left( E(g|x_1, \ldots, x_k, \cdot) \right)$$

   local

3. **Go on until you get to the root node $\Box$, where:**

   $$E(g|\Box) = E(g).$$
The first probability tree?

Christiaan Huygens, *Van Rekeningh in Spelen van Geluck* (1656–1657)
Major restrictive assumption

Until now, we have assumed that we have sufficient information in order to specify, in each node $s$, a probability mass function $p_s$ on the set $\mathcal{X}$ of possible values for the next state.

More general uncertainty models

We consider credal sets as more general uncertainty models: closed convex subsets of $\Sigma_\mathcal{X}$.
Imprecise probability trees
Definition and interpretation

**Definition**

An *imprecise probability tree* is a probability tree where in each node $s$ the local uncertainty model is an imprecise probability model $\mathcal{M}_s$, or equivalently, its associated upper expectation $\overline{E}_s$:

$$\overline{E}_s(f) = \max \{ E_p(f) : p \in \mathcal{M}_s \} \text{ for all real maps } f \text{ on } \mathcal{X}.$$
Imprecise probability trees
Definition and interpretation

An imprecise probability tree can be seen as an infinity of compatible precise probability trees: choose in each node a probability mass function $p_s$ from the set $M_s$.}

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Imprecise probability trees
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\overline{E}_s(f) = \max \left\{ E_p(f) : p \in \mathcal{M}_s \right\}
\]

for all real maps \( f \) on \( \mathcal{X} \).

An imprecise probability tree can be seen as an infinity of compatible precise probability trees: choose in each node \( s \) a probability mass function \( p_s \) from the set \( \mathcal{M}_s \).
Imprecise probability trees

Definition and interpretation

An imprecise probability tree can be seen as an infinity of compatible precise probability trees: choose in each node a probability mass function $p_s$ from the set $M_s$. 

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Imprecise Markov chains
Imprecise probability trees
Associated lower and upper expectations

For each real map $g = g(X_1, \ldots, X_n)$, each node $s = (x_1, \ldots, x_k)$, and each such compatible precise probability tree, we can calculate the expectation

$$E(g|x_1, \ldots, x_k)$$

using the backwards recursion method described before.

By varying over each compatible probability tree, we get a closed real interval:

$$[E(g|x_1, \ldots, x_k), \bar{E}(g|x_1, \ldots, x_k)]$$

We want a better, more efficient method to calculate these lower and upper expectations $\underline{E}(g|x_1, \ldots x_k)$ and $\bar{E}(g|x_1, \ldots, x_k)$. 
Theorem (Law of Iterated Expectation)

Suppose we know \( \overline{E}(g|s,x) \) for all \( x \in \mathcal{X} \), then we can calculate \( \overline{E}(g|s) \) by backwards recursion using the local model \( \overline{E}_s \):

\[
\overline{E}(g|s) = \overline{E}_s(\overline{E}(g|s,\cdot)) = \max_{p_s \in \mathcal{M}_s} \sum_{x \in \mathcal{X}} p_s(x) \overline{E}(g|s,x).
\]

The complexity of calculating the \( \overline{E}(g|s) \), as a function of \( n \), is therefore essentially the same as in the precise case!
Precise Markov chains

Definition

The uncertain process is a stationary precise Markov chain when all $M_s$ are singletons (precise), and

1. $M_{\square} = \{m_1\}$,

2. the Markov Condition is satisfied:

$$M(x_1, \ldots, x_n) = \{q(\cdot | x_n)\}.$$
Precise Markov chains

Definition

For each $x \in X$, the transition mass function $q(\cdot|x)$ corresponds to an expectation operator:

$$E(f|x) = \sum_{z \in X} q(z|x) f(z).$$
Precise Markov chains

Definition

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1. $M_{\square} = \{m_1\}$,

2. the Markov Condition is satisfied:

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For each $x \in \mathbb{X}$, the transition mass function $q(\cdot | x)$ corresponds to an expectation operator:

$$E(f | x) = \sum_{z \in \mathbb{X}} q(z | x) f(z).$$
Consider the linear transformation $T$ of $\mathcal{L}(\mathcal{X})$, called transition operator:

$$T: \mathcal{L}(\mathcal{X}) \rightarrow \mathcal{L}(\mathcal{X}): f \mapsto Tf$$

where $Tf$ is the real map given by, for any $x \in \mathcal{X}$:

$$Tf(x) := E(f|x) = \sum_{z \in \mathcal{X}} q(z|x)f(z)$$

$T$ is the dual of the linear transformation with Markov matrix $M$, with elements $M_{xy} := q(y|x)$. 
Precise Markov chains

Transition operators

**Definition**

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$T$ is the dual of the linear transformation with Markov matrix $M$, with elements $M_{xy} := q(y|x)$.

Then the Law of Iterated Expectation yields:

$$E_n(f) = E_1(T^{n-1}f), \text{ and dually, } m_n = M^{n-1}m_1.$$ 

Complexity is linear in the number of time steps $n$. 

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Imprecise Markov chains

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Imprecise Markov chains

Definition

The uncertain process is a stationary imprecise Markov chain when the Markov Condition is satisfied:

\[ M(x_1, \ldots, x_n) = Q(\cdot | x_n). \]
Imprecise Markov chains

Definition

An imprecise Markov chain can be seen as an infinity of probability trees.

For each $x \in X$, the local transition model $Q(\cdot|x)$ corresponds to lower and upper expectation operators:

$$E(f|x) = \min_{p \in Q(\cdot|x)} E_p(f)$$

$$E(f|x) = \max_{p \in Q(\cdot|x)} E_p(f).$$
**Imprecise Markov chains**

**Definition**

The uncertain process is a stationary *imprecise Markov chain* when the **Markov Condition** is satisfied:

\[ \mathcal{M}(x_1, \ldots, x_n) = \mathcal{Z}(\cdot | x_n). \]

An imprecise Markov chain can be seen as an infinity of probability trees.

For each \( x \in \mathcal{X} \), the local transition model \( \mathcal{Z}(\cdot | x) \) corresponds to **lower** and **upper** expectation operators:

\[
E(f | x) = \min \{ E_p(f) : p \in \mathcal{Z}(\cdot | x) \} \\
\bar{E}(f | x) = \max \{ E_p(f) : p \in \mathcal{Z}(\cdot | x) \}.
\]
Consider the non-linear transformations $T$ and $\overline{T}$ of $\mathcal{L}(\mathcal{X})$, called lower and upper transition operators:

$T : \mathcal{L}(\mathcal{X}) \rightarrow \mathcal{L}(\mathcal{X}) : f \mapsto Tf$

$\overline{T} : \mathcal{L}(\mathcal{X}) \rightarrow \mathcal{L}(\mathcal{X}) : f \mapsto \overline{T}f$

where the real maps $Tf$ and $\overline{T}f$ are given by:

$Tf(x) := E(f | x) = \min \left\{ E_p(f) : p \in \mathcal{Q} (\cdot | x) \right\}$

$\overline{T}f(x) := \overline{E}(f | x) = \max \left\{ E_p(f) : p \in \mathcal{Q} (\cdot | x) \right\}$
Imprecise Markov chains
Lower and upper transition operators

**Definition**

Consider the non-linear transformations $T$ and $\overline{T}$ of $\mathcal{L}(\mathcal{X}^*)$, called lower and upper transition operators:

$$T: \mathcal{L}(\mathcal{X}^*) \rightarrow \mathcal{L}(\mathcal{X}^*): f \mapsto T f$$

$$\overline{T}: \mathcal{L}(\mathcal{X}^*) \rightarrow \mathcal{L}(\mathcal{X}^*): f \mapsto \overline{T} f$$

where the real maps $T f$ and $\overline{T} f$ are given by:

$$T f(x) := E(f|x) = \min \{ E_p(f) : p \in \mathcal{D}(\cdot|x) \}$$

$$\overline{T} f(x) := \overline{E}(f|x) = \max \{ E_p(f) : p \in \mathcal{D}(\cdot|x) \}$$

Then the Law of Iterated Expectation yields:

$$E_n(f) = E_1(T^{n-1} f) \quad \text{and} \quad \overline{E}_n(f) = \overline{E}_1(\overline{T}^{n-1} f).$$

Complexity is still linear in the number of time steps $n$. 

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Random Walks

An example with a two-element state space

Consider a \textbf{two-element} state space:

\[ \mathcal{X} = \{a, b\}, \]

with \textbf{upper expectation} \( \overline{E}_1 \) for the first state, and for each \((x_1, \ldots, x_n) \in \{a, b\}^n\), with \( \varepsilon \in [0, 1] \),

\[ M_{(x_1, \ldots, x_n)} = M_{x_n} = (1 - \varepsilon) \{q(\cdot | x_n)\} + \varepsilon \Sigma_{\{a,b\}} \]

or in other words, for the \textbf{upper transition operator}

\[ \overline{T} = (1 - \varepsilon)T + \varepsilon \max \]

where \( T \) is the \textbf{linear transition operator} determined by

\[ M := \begin{bmatrix} TI_{\{a\}}(a) & TI_{\{b\}}(a) \\ TI_{\{a\}}(b) & TI_{\{b\}}(b) \end{bmatrix} = \begin{bmatrix} q(a|a) & q(b|a) \\ q(a|b) & q(b|b) \end{bmatrix}. \]
Random Walks
Stationary distribution

It is a matter of simple verification that for \( n \geq 1 \) and \( f \in \mathcal{L}(\mathcal{X}) \):

\[
\overline{T}^n f = (1 - \varepsilon)^n T^n f + \varepsilon \sum_{k=0}^{n-1} (1 - \varepsilon)^k \max T^k f,
\]

and therefore, using the Law of Iterated Expectation,

\[
\overline{E}_{n+1}(f) = \overline{E}_1(\overline{T}^n f) = (1 - \varepsilon)^n \overline{E}_1(T^n f) + \varepsilon \sum_{k=0}^{n-1} (1 - \varepsilon)^k \max T^k f.
\]

If we now let \( n \to \infty \), we see that the limit exists and is independent of the initial upper expectation \( \overline{E}_1 \):

\[
\overline{E}_\infty(f) = \varepsilon \sum_{k=0}^{\infty} (1 - \varepsilon)^k \max T^k f.
\]
Contaminated Random Walk

When

\[ Tf(a) = Tf(b) = \frac{1}{2}[f(a) + f(b)], \text{ i.e., } M = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix} \]

then we find that

\[ \overline{E}_\infty(f) = (1 - \varepsilon)\frac{1}{2}[f(a) + f(b)] + \varepsilon \max f. \]
Contaminated Cycle

When

\[ T(f(a)) = f(b) \quad \text{and} \quad T(f(b)) = f(a), \quad \text{i.e.,} \quad M = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \]

then we find that

\[ E_\infty(f) = \max f. \]
Lower and upper mass functions

Another example with \( \mathcal{X} = \{a, b, c\} \)

\[
\begin{bmatrix}
TI_{\{a\}} & TI_{\{b\}} & TI_{\{c\}}
\end{bmatrix}
= \begin{bmatrix}
q(a|a) & q(b|a) & q(c|a) \\
q(a|b) & q(b|b) & q(c|b) \\
q(a|c) & q(b|c) & q(c|c)
\end{bmatrix}
= \frac{1}{200}
\begin{bmatrix}
9 & 9 & 162 \\
144 & 18 & 18 \\
9 & 162 & 9
\end{bmatrix}
\]

\[
\begin{bmatrix}
\bar{TI}_{\{a\}} & \bar{TI}_{\{b\}} & \bar{TI}_{\{c\}}
\end{bmatrix}
= \begin{bmatrix}
\bar{q}(a|a) & \bar{q}(b|a) & \bar{q}(c|a) \\
\bar{q}(a|b) & \bar{q}(b|b) & \bar{q}(c|b) \\
\bar{q}(a|c) & \bar{q}(b|c) & \bar{q}(c|c)
\end{bmatrix}
= \frac{1}{200}
\begin{bmatrix}
19 & 19 & 172 \\
154 & 28 & 28 \\
19 & 172 & 19
\end{bmatrix}
\]
Lower and upper mass functions

Another example with $\mathcal{X} = \{a, b, c\}$

$n = 1$

$n = 2$

$n = 3$

$n = 4$

$n = 5$

$n = 6$

$n = 7$

$n = 8$

$n = 9$

$n = 10$

$n = 22$

$n = 1000$
Consider a stationary imprecise Markov chain with finite state set $\mathcal{X}$ and an upper transition operator $\overline{T}$. Suppose that $\overline{T}$ is regular, meaning that there is some $n > 0$ such that $\min_{\mathcal{X}} \overline{T}^n I_{\{x\}} > 0$ for all $x \in \mathcal{X}$. Then for every initial upper expectation $\overline{E}_1$, the upper expectation $\overline{E}_n = \overline{E}_1 \circ \overline{T}^{n-1}$ for the state at time $n$ converges point-wise to the same upper expectation $\overline{E}_\infty$:

$$\lim_{n \to \infty} \overline{E}_n(h) = \lim_{n \to \infty} \overline{E}_1(\overline{T}^{n-1} h) := \overline{E}_\infty(h)$$

for all $h$ in $\mathcal{L}(\mathcal{X})$. Moreover, the corresponding limit upper expectation $\overline{E}_\infty$ is the only $\overline{T}$-invariant upper expectation on $\mathcal{L}(\mathcal{X})$, meaning that $\overline{E}_\infty = \overline{E}_\infty \circ \overline{T}$. 

**Theorem (De Cooman, Hermans and Quaeghebeur, 2008)**
Mean first passage times

Definition

Let the random process $\tau_{xy}$ be the first time $n > 0$ such that $X_{n+1} = y$, if the process starts out in $X_1 = x$.

We are interested in the lower and upper mean first passage times:

$$M_{xy} = E(\tau_{xy} | x) \quad \text{and} \quad \overline{M}_{xy} = \overline{E}(\tau_{xy} | x).$$

If $x = y$, we call

$$R_x := M_{xx} = E(\tau_{xx} | x) \quad \text{and} \quad \overline{R}_x := \overline{M}_{xx} = \overline{E}(\tau_{xx} | x)$$

lower and upper mean recurrence times.
Mean first passage times
Non-linear equations for mean first passage times

Now for any trajectory \((x, x_2, x_3, \ldots)\) starting in \(x\):

\[
\tau_{xy}(x, x_2, x_3, \ldots) = \begin{cases} 
1 & ; \quad x_2 = y \\
1 + \tau_{x_2y}(x_2, x_3, \ldots) & ; \quad x_2 \neq y
\end{cases}
\]

which is a recursive relation, so if we use the Law of Iterated Expectation, stationarity and the Markov Property, we are led to the non-linear equations:

\[
\bar{M}_y = 1 + T[(1 - \delta_y)\bar{M}_y] \quad \text{and} \quad \underline{M}_y = 1 + \underline{T}[(1 - \delta_y)\underline{M}_y].
\]
Mean first passage times

Examples

We find after solving the non-linear equations that:

Contaminated random walk

\[
\begin{align*}
R_a &= R_b = M_{ab} = M_{ba} = \frac{2}{1 + \varepsilon} \\
\bar{R}_a &= \bar{R}_b = \bar{M}_{ab} = \bar{M}_{ba} = \frac{2}{1 - \varepsilon}.
\end{align*}
\]

Contaminated cycle

\[
\begin{align*}
R_a &= R_b = 2 - \varepsilon \quad \text{and} \quad M_{ab} = M_{ba} = 1 \\
\bar{R}_a &= \bar{R}_b = \frac{2 - \varepsilon}{1 - \varepsilon} \quad \text{and} \quad \bar{M}_{ab} = \bar{M}_{ba} = \frac{1}{1 - \varepsilon}.
\end{align*}
\]

