Independent natural extension for sets of desirable gambles

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Sets of desirable gambles

Consider a variable $X$ assuming values in $\mathcal{X}$. A gamble on $X$ is a map $f : \mathcal{X} \to \mathbb{R}$. The set of all gambles on $X$ is $\mathbb{G}(X)$. A set of desirable gambles $\mathcal{G} \subseteq \mathbb{G}(X)$ is a model for the gambles that a subject strictly accepts.

Definition 1. A set of desirable gambles $\mathcal{G} \subseteq \mathbb{G}(X)$ is called coherent if:

1. $\mathcal{G}(\emptyset) = \{\emptyset\}$ (avoiding non-positivity);
2. $\mathcal{G}(\emptyset)$ is a partial order (accepting partial gains);
3. $\mathcal{G}$ is a model for the gambles that a subject strictly accepts.

We denote by $\mathcal{D}$ the set of all coherent sets of desirable gambles on $X$.

Theorem 1 (Natural extension). Let $\mathcal{G}$ be any subset of $\mathcal{D}$. Then

$$\mathcal{G}(\mathcal{G}) : \mathcal{G} \subseteq \mathcal{D} \mapsto \bigcap \{ \mathcal{G} \subseteq \mathcal{D} : \mathcal{G} \subseteq \mathcal{G} \}$$

is the smallest coherent set of desirable gambles that includes $\mathcal{G}$.

Theorem 2. $\mathcal{G}(\mathcal{G})$ avoids non-positivity if and only if $\mathcal{G}$ is maximal.

Moreover, its natural extension is given by:

$$\mathcal{G}(\mathcal{G}) = \bigcap \{ \mathcal{G} \subseteq \mathcal{D} : \mathcal{G} \subseteq \mathcal{G} \}.$$ 

Maximal sets of desirable gambles

An element $\mathcal{G}$ of $\mathcal{D}$ is called maximal if it is not strictly included in any other element of $\mathcal{D}$.

Let $\mathcal{M}$ denote the set of all maximal elements of $\mathcal{D}$. Note that $\mathcal{G}$ is a maximal coherent set of desirable gambles if and only if

$$\mathcal{G} \subseteq \mathcal{M}.$$ 

Theorem 3. $\mathcal{M}$ avoids non-positivity if and only if $\mathcal{G}$ is maximal.

Moreover, its natural extension is given by:

$$\mathcal{G}(\mathcal{M}) = \bigcap \{ \mathcal{G} \subseteq \mathcal{D} : \mathcal{G} \subseteq \mathcal{G} \}.$$ 

More than one variable

Consider a finite number of variables $X_n$, $n \in \mathbb{N}$, in the respective finite sets $\mathcal{X}_n$.

For $R \subseteq \mathcal{X}$, we denote by $X_R$ the tuple of variables that takes values in the Cartesian product $X_R = \times_{n \in R} X_n$.

We denote by $\mathbb{G}(X_R)$ the set of gambles on $X_R$.

Suppose $\mathcal{G}_R \subseteq \mathbb{G}(X_R)$ models a subject's beliefs about $X_R$.

Marginalisation. The corresponding beliefs about the variable $X_n$, where $O \subseteq \mathbb{N}$, are given by the marginal model:

$$\mathbb{G}(\mathcal{G}_R) = \{ g \in \mathbb{G}(X_O) : g \in \mathbb{G}(X_R) \}.$$ 

Conditioning. Conditioning the model $\mathcal{G}_R$ with the information that $X_i = \omega_i$, with $I \subseteq O$, leads to the updated model:

$$\mathcal{G}_R\{X_I\} = \{ g \in \mathbb{G}(X_O) : I_{\omega} g \in \mathbb{G}(X_O) \}.$$ 

Theorem 3. Coherence is preserved under marginalisation and conditioning.

Independent natural extension

Consider, for any subset $I$ of $O$ and any $\omega \in \mathcal{X}_I$, the set of desirable gambles on $X_O$,

$$\mathcal{G}(\omega \in \mathcal{G}(X_I)) = \mathbb{G}(X_O) \setminus \mathcal{G}(\omega \in \mathcal{G}(X_I)).$$ 

We use these sets to construct the following set of desirable gambles on $X_O$:

$$\mathcal{G}(\omega \in \mathcal{G}(X_I)) = \bigcap \{ \mathcal{G} \subseteq \mathcal{D} : \mathcal{G} \subseteq \mathcal{G}(\omega \in \mathcal{G}(X_I)) \}.$$ 

Constructing joints from marginals. Suppose we have coherent sets $\mathcal{G}_n$ of desirable gambles on $X_n$, for each $n \in \mathbb{N}$.

Definition 3. We call independent product of the $\mathcal{G}_n$ any independent set of desirable gambles $\mathcal{G}_n \in \mathcal{D}(X_n)$ that marginalises to the $\mathcal{G}_n$ for all $n \in \mathbb{N}$:

$$\mathbb{G}(\mathcal{G}_n) = \mathcal{G}_n$$

for all $n \in \mathbb{N}$.

There are usually infinitely many such independent products.

We are looking for the smallest one: the independent natural extension of the $\mathcal{G}_n$.

Strong product

We define the strong product $\mathbb{G}(\mathcal{G}_n)$ as the set of desirable gambles on the product space $X_O$ given by:

$$\mathbb{G}(\mathcal{G}_n) = \bigcap \{ \mathcal{G} \subseteq \mathcal{D}(X_O) : \mathcal{G} \subseteq \mathcal{G}_n \}.$$ 

Observe that for maximal sets $\mathcal{G}_n \in \mathcal{M}(X_n)$, $n \in \mathbb{N}$ the strong product and the independent natural extension coincide:

$$\mathbb{G}(\mathcal{G}_n) = \mathbb{G}(\mathcal{G}_n).$$ 

Proposition 8 (Marginalisation). Let $I$ be any subset of $O$, then

$$\mathbb{G}(\omega \in \mathcal{G}(X_I)) = \mathbb{G}(\mathcal{G}(X_I)) \cap \mathbb{G}(\mathcal{G}(X_I)).$$ 

Proposition 9 (Conditioning). The strong product $\mathbb{G}(\mathcal{G}_n)$ is independent: for all disjoint subsets $I$ and $O$ of $N$, and all $\omega \in \mathcal{X}_O$,

$$\mathbb{G}(\omega \in \mathcal{G}(X_I)) \subseteq \mathbb{G}(\mathcal{G}(X_I)) \cap \mathbb{G}(\mathcal{G}(X_I)).$$ 

It is still an open problem at this point whether, like the natural extension, the strong product is associative.