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Representing and assessing exchangeable lower previsions

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1 Introduction

This paper deals with belief models for both (finite) collections and (infinite) sequences of exchangeable random variables taking a finite number of values. When such collections or sequences are assumed to be exchangeable, this more or less means that their specific order is irrelevant. One of the reasons why exchangeability is deemed important, especially by Bayesians, is that, by virtue of de Finetti's Representation Theorem, an exchangeable model can be seen as a convex mixture of multinomial models. This has given some ground (de Finetti, 1937, 1974-1975; Dawid, 1985) to the claim that aleatory probabilities and IID processes can be eliminated from statistics, and that we can restrict ourselves to exchangeable sequences instead.

De Finetti presented his study of exchangeability in terms of the behavioural notion of previsions, or fair prices. One assumption underlying his approach is that a subject should be able to specify a fair price $P(f)$ for any risky transaction, or *gamble*, f. He should therefore always be willing and able to decide, for any real number r, between selling the gamble f for r, or buying it for that price. This may not always be realistic, and it has been suggested that we should explicitly allow for a subject's indecision, by distinguishing between his *lower prevision* $\underline{P}(f)$, which is the supremum price for which he is willing to buy the gamble f, and his *upper prevision* $\overline{P}(f)$, which is the infimum price for which he is willing to sell f. The resulting *theory of coherent lower previsions*, brilliantly defended by Walley (1991), generalises de Finetti's behavioural account of subjective, epistemic probability, and tries to make it more realistic by allowing for a subject's indecision.

But, even if we allow for a subject's indecision by considering lower, rather than precise, previsions, it remains very useful to consider what are the consequences of a subject's exchangeability assessment. This is our motivation for studying exchangeable lower previsions here. The exchangeability assessment has a clear impact on the structure of so-called *exchangeable* coherent lower previsions. We show how they can be written as a combination of (i) a coherent prevision expressing that permutations of realisations of such collections or sequences are considered equally likely, and (ii) a coherent lower prevision for the 'frequency' of occurrence of the different values the random variables can take. This is the essence of representation in de Finetti's sense: we generalise his results to coherent lower previsions. We also solve a more practical problem: how to extend a number of lower prevision assessments to an exchangeable lower prevision that is as conservative as possible.

2 Exchangeable random variables

Consider $N \geq 1$ random variables $X_1, \ldots, X_N$ taking values in a non-empty and finite set $\mathscr{X}$. A subject's beliefs about the values that these random variables $\mathbf{X} = (X_1, \ldots, X_N)$ assume jointly in $\mathscr{X}^N$ is given by their (joint) distribution, which is a coherent lower prevision $\underline{P}^N_{\mathscr{X}}$ defined on the set $\mathscr{L}(\mathscr{X}^N)$ of all gambles (bounded random variables) on $\mathscr{X}^N$. Let us denote by $\mathscr{P}_N$ the set of all permutations of $\{1,\dots,N\}$. With any such permutation π we can associate, by the procedure of lifting, a permutation of $\mathscr{X}^N$, also denoted by π, that maps any $\mathbf{x} = (x_1,\dots,x_N)$ in $\mathscr{X}^N$ to $\pi\mathbf{x} := (x_{\pi(1)},\dots,x_{\pi(N)})$. Similarly, with any gamble f on $\mathscr{X}^N$, we can consider the permuted gamble $\pi f := f \circ \pi$, or in other words, $(\pi f)(\mathbf{x}) = f(\pi\mathbf{x})$ for all $\mathbf{x} \in \mathscr{X}^N$.

Now a subject judges the random variables $X_1, \dots, X_N$ to be *exchangeable* when he is disposed to exchange any gamble f for the permuted gamble πf, meaning that $\underline{P}^N_{\mathscr{X}}(\pi f - f) \geq 0$, for any permutation π (Walley, 1991). Taking into account the properties of coherence (a rationality criterion, see Walley (1991)), this means that $\underline{P}^N_{\mathscr{X}}(\pi f - f) = \underline{P}^N_{\mathscr{X}}(f - \pi f) = 0$ for all gambles f on $\mathscr{X}^N$ and all permutations π in $\mathscr{P}_N$. In this case, we also call the joint lower prevision $\underline{P}^N_{\mathscr{X}}$ *exchangeable*. A subject will make an assumption of exchangeability when there is evidence that the processes generating the values of the random variables are (physically) similar (Walley, 1991, Section 9.5.2), and consequently the order in which the variables are observed is not important.

When $\underline{P}^N_{\mathscr{X}}$ is in particular a coherent (precise) prevision $P^N_{\mathscr{X}}$, exchangeability is equivalent to having $P^N_{\mathscr{X}}(\pi f) = P^N_{\mathscr{X}}(f)$ for all gambles f and all permutations π, and this is also equivalent to having $p^N_{\mathscr{X}}(\mathbf{x}) = p^N_{\mathscr{X}}(\pi\mathbf{x})$ for all $\mathbf{x}$ in $\mathscr{X}^N$, where $p^N_{\mathscr{X}}$ is the probability mass function of $P^N_{\mathscr{X}}$, defined by $p^N_{\mathscr{X}}(\mathbf{x}) = P^N_{\mathscr{X}}(\{\mathbf{x}\})$. This is essentially de Finetti's (1937) definition for the exchangeability of a prevision. A useful relation exists between exchangeable coherent previsions and exchangeable coherent lower previsions:

Proposition 1. *Let $\underline{P}^N_{\mathscr{X}}$ be the lower envelope of some set of coherent previsions $\mathscr{M}^N_{\mathscr{X}}$. Then $\underline{P}^N_{\mathscr{X}}$ is exchangeable if and only if all the coherent previsions $P^N_{\mathscr{X}}$ in $\mathscr{M}^N_{\mathscr{X}}$ are exchangeable.*

Exchangeable coherent lower previsions have a very simple representation, in terms of sampling without replacement. Consider any $\mathbf{x} \in \mathscr{X}^N$, then the so-called *invariant atom* $[\mathbf{x}] = \{\pi\mathbf{x} \colon \pi \in \mathscr{P}_N\}$ is the smallest non-empty subset of $\mathscr{X}^N$ that contains $\mathbf{x}$ and that is invariant under all permutations π in $\mathscr{P}_N$. We can characterise these invariant atoms using the *counting maps* $T^N_x \colon \mathscr{X}^N \to \mathbb{N}_0$ defined for all $x \in \mathscr{X}$ in such a way that $T^N_x(\mathbf{z}) = T^N_x(z_1,\dots,z_N) := |\{k \in \{1,\dots,N\} \colon z_k = x\}|$, is the number of components of the N-tuple $\mathbf{z}$ that assume the value x. We denote by $\mathbf{T}^N_{\mathscr{X}}$ the vector-valued map from $\mathscr{X}^N$ to $\mathbb{N}_0^{\mathscr{X}}$ whose components are the T^N_x, $x \in \mathscr{X}$. Observe that $\mathbf{T}^N_{\mathscr{X}}$ actually assumes values in the set of *count vectors* $\mathscr{N}^N_{\mathscr{X}} := \left\{\mathbf{m} \in \mathbb{N}_0^{\mathscr{X}} \colon \sum_{x\in\mathscr{X}} m_x = N\right\}$. Since permuting the components of a vector leaves the counts invariant, for all $\mathbf{y}$ and $\mathbf{z}$ in $\mathscr{X}^N$ it holds that $\mathbf{y} \in [\mathbf{z}]$ iff $\mathbf{T}^N_{\mathscr{X}}(\mathbf{y}) = \mathbf{T}^N_{\mathscr{X}}(\mathbf{z})$. If $\mathbf{T}^N_{\mathscr{X}}(\mathbf{x}) = \mathbf{m}$, then we denote $[\mathbf{x}]$ also by $[\mathbf{m}]$. It has $\nu(\mathbf{m}) := \binom{N}{\mathbf{m}} = N!/\prod_{x\in\mathscr{X}} m_x!$ elements.

If the joint random variable $\mathbf{X} = (X_1,\dots,X_N)$ assumes the value $\mathbf{z}$ in $\mathscr{X}^N$, then the corresponding count vector assumes the value $\mathbf{T}^N_{\mathscr{X}}(\mathbf{z})$ in $\mathscr{N}^N_{\mathscr{X}}$. This means that we can see $\mathbf{T}^N_{\mathscr{X}}(\mathbf{X}) = \mathbf{T}^N_{\mathscr{X}}(X_1,\dots,X_N)$ as a random variable in $\mathscr{N}^N_{\mathscr{X}}$. If the available information about the values that $\mathbf{X}$ assumes in $\mathscr{X}^N$ is given by the coherent exchangeable lower prevision $\underline{P}^N_{\mathscr{X}}$, then the corresponding uncertainty model for the values that $\mathbf{T}^N_{\mathscr{X}}(\mathbf{X})$ assumes in $\mathscr{N}^N_{\mathscr{X}}$ is given by the coherent *induced* lower prevision $\underline{Q}^N_{\mathscr{X}}$ on $\mathscr{L}(\mathscr{N}^N_{\mathscr{X}})$, given by $\underline{Q}^N_{\mathscr{X}}(h) := \underline{P}^N_{\mathscr{X}}(h \circ \mathbf{T}^N_{\mathscr{X}}) = \underline{P}^N_{\mathscr{X}}\left(\sum_{\mathbf{m}\in\mathscr{N}^N_{\mathscr{X}}} h(\mathbf{m}) I_{[\mathbf{m}]}\right)$ for all gambles h on $\mathscr{N}^N_{\mathscr{X}}$. Any exchangeable coherent lower prevision $\underline{P}^N_{\mathscr{X}}$ is in fact *completely determined* by the corresponding distribution $\underline{Q}^N_{\mathscr{X}}$ of the count vectors.

Theorem 2 (Representation theorem for exchangeable variables)**.** *Let $N \geq 1$ and let $\underline{P}^N_{\mathscr{X}}$ be a coherent exchangeable lower prevision on $\mathscr{L}(\mathscr{X}^N)$. Let f be any gamble on $\mathscr{X}^N$. Then the following statements hold:*

1. *$\underline{P}^N_{\mathscr{X}}(f) = \underline{P}^N_{\mathscr{X}}(\hat{f})$ where $\hat{f}$ is the gamble on $\mathscr{X}^N$ that is constant on the permutation invariant atoms of $\mathscr{X}^N$, and given by $\hat{f}(\mathbf{z}) = MuHy^N_{\mathscr{X}}(f|\mathbf{m}) := \frac{1}{\nu(\mathbf{m})}\sum_{\mathbf{z}\in[\mathbf{m}]} f(\mathbf{z})$ for all $\mathbf{z} \in [\mathbf{m}]$, where $\mathbf{m} \in \mathscr{N}^N_{\mathscr{X}}$;*

2. *$\underline{P}^N_{\mathscr{X}}(f) = \underline{Q}^N_{\mathscr{X}}\left(MuHy^N_{\mathscr{X}}(f|\cdot)\right)$, where $MuHy^N_{\mathscr{X}}(f|\cdot)$ is the gamble on $\mathscr{N}^N_{\mathscr{X}}$ that takes the value $MuHy^N_{\mathscr{X}}(f|\mathbf{m})$ in $\mathbf{m} \in \mathscr{N}^N_{\mathscr{X}}$.*

Consequently, a lower prevision on $\mathscr{L}(\mathscr{X}^N)$ is exchangeable if and only if it has the form $\underline{Q}\big(MuHy^N_{\mathscr{X}}(\cdot|\cdot)\big)$, where $\underline{Q}$ is any coherent lower prevision on $\mathscr{L}(\mathscr{N}^N_{\mathscr{X}})$.

As $MuHy^N_{\mathscr{X}}(f|\mathbf{m})$ is the prevision associated with the multiple hypergeometric distribution, this theorem implies that any collection of N exchangeable random variables in $\mathscr{X}$ can be seen as the result of N random drawings from an urn with N balls whose types are characterised by the elements x of $\mathscr{X}$, whose composition $\mathbf{m}$ is unknown, but for which the available information about the composition is modelled by a coherent lower prevision on $\mathscr{L}(\mathscr{N}_{\mathscr{X}}^{N})$. That exchangeable coherent previsions can be interpreted in terms of sampling without replacement from an urn with unknown composition, is actually well-known, and essentially goes back to de Finetti's (1937) work on exchangeability. Our result for the more general case of exchangeable coherent *lower* previsions and random variables that may assume more than two values is a special case of a much more general representation theorem (De Cooman and Miranda, 2007).

3 Exchangeable sequences

Consider now a sequence $X_1, \ldots, X_n, \ldots$ of random variables taking values in the same non-empty set $\mathscr{X}$. Then this sequence is called *exchangeable* if any finite collection of random variables taken from this sequence is exchangeable. We can regard the exchangeable sequence as a single random variable $\mathbf{X}$ assuming values in the set $\mathscr{X}^{\mathbb{N}}$, where $\mathbb{N}$ is the set of the natural numbers. Its possible values $\mathbf{x}$ are sequences $x_1, \ldots, x_n, \ldots$ of elements of $\mathscr{X}$. We can model the available information about the value that $\mathbf{X}$ assumes in $\mathscr{X}^{\mathbb{N}}$ by a coherent lower prevision $\underline{P}_{\mathscr{X}}^{\mathbb{N}}$ on $\mathscr{L}(\mathscr{X}^{\mathbb{N}})$, called the *distribution* of the exchangeable random sequence $\mathbf{X}$. The random sequence $\mathbf{X}$, or its distribution $\underline{P}_{\mathscr{X}}^{\mathbb{N}}$, is exchangeable if and only if all its $\mathscr{X}^n$*-marginals* $\underline{P}_{\mathscr{X}}^{n}$ are exchangeable for $n \geq 1$, where for any gamble f on $\mathscr{X}^n$ it holds that $\underline{P}_{\mathscr{X}}^{n}(f) = \underline{P}_{\mathscr{X}}^{\mathbb{N}}(\widetilde{f})$, where $\widetilde{f}$ is the cylindrical extension of f to $\mathscr{X}^{\mathbb{N}}$: for all $\mathbf{x} \in \mathscr{X}^{\mathbb{N}}$, $\widetilde{f}(\mathbf{x}) := f(x_1, \ldots, x_n)$. In addition, the family of exchangeable coherent lower previsions $\underline{P}_{\mathscr{X}}^{n}$, $n \geq 1$, satisfies the following '*time consistency*' requirement: $\underline{P}_{\mathscr{X}}^{n}(f) = \underline{P}_{\mathscr{X}}^{n+k}(\widetilde{f})$, for all $n \geq 1$, $k \geq 0$, and all gambles f on $\mathscr{X}^n$, where $\widetilde{f}$ now denotes the cylindrical extension of f to $\mathscr{X}^{n+k}$: $\underline{P}_{\mathscr{X}}^{n}$ should be the $\mathscr{X}^n$-marginal of any $\underline{P}_{\mathscr{X}}^{n+k}$. In terms of the distributions of the count vectors, the time consistency requirement means that $\underline{Q}_{\mathscr{X}}^{n}(h) = \underline{Q}_{\mathscr{X}}^{n+k}\big(\sum_{\mathbf{m} \in \mathscr{N}_{\mathscr{X}}^{n}} \frac{\nu(\cdot - \mathbf{m})\nu(\mathbf{m})}{\nu(\cdot)} h(\mathbf{m})\big)$ for all $n \geq 1$, $k \geq 0$ and $h \in \mathscr{L}(\mathscr{N}_{\mathscr{X}}^{n})$.

Any collection of $n \geq 1$ random variables taken from such an exchangeable sequence has the same distribution as the first n variables $X_1, \ldots, X_n$, which is the exchangeable coherent lower prevision $\underline{P}_{\mathscr{X}}^{n}$ on $\mathscr{L}(\mathscr{X}^n)$. Conversely, suppose we have a collection of exchangeable coherent lower previsions $\underline{P}_{\mathscr{X}}^{n}$ on $\mathscr{L}(\mathscr{X}^n)$, $n \geq 1$ that satisfy the time consistency requirement. Then any coherent lower prevision $\underline{P}_{\mathscr{X}}^{\mathbb{N}}$ on $\mathscr{L}(\mathscr{X}^{\mathbb{N}})$ that has $\mathscr{X}^n$-marginals $\underline{P}_{\mathscr{X}}^{n}$ is exchangeable. The smallest, or most conservative such (exchangeable) coherent lower prevision is given by $\underline{E}_{\mathscr{X}}^{\mathbb{N}}(f) := \sup_{n \in \mathbb{N}} \underline{P}_{\mathscr{X}}^{n}\big(\underline{\mathrm{proj}}_n(f)\big)$, where f is any gamble on $\mathscr{X}^{\mathbb{N}}$, and its *lower projection* $\underline{\mathrm{proj}}_n(f)$ on $\mathscr{X}^n$ is the gamble on $\mathscr{X}^n$ that is defined by $\underline{\mathrm{proj}}_n(f)(\mathbf{x}) := \inf_{z_k = x_k, k=1,\ldots,n} f(\mathbf{z})$ for all $\mathbf{x} \in \mathscr{X}^n$.

De Finetti (1937 and 1974-1975) has proven a representation result for exchangeable sequences with coherent previsions that generalises Theorem 2, and where multinomial distributions take over the rôle that the multiple hypergeometric ones play for finite collections of exchangeable variables. Here, we present another, arguably even simpler, way to prove the same results, which is moreover valid for coherent lower previsions. Consider a sequence of IID random variables $Y_1, \ldots, Y_n$, with probability mass function $\boldsymbol{\theta}$, where $\boldsymbol{\theta}$ is an element of the $\mathscr{X}$*-simplex* $\Sigma_{\mathscr{X}} = \big\{\boldsymbol{\theta} \in \mathbb{R}^{\mathscr{X}} : (\forall x \in \mathscr{X})(\theta_x \geq 0) \text{ and } \sum_{x \in \mathscr{X}} \theta_x = 1\big\}$. The distribution of $(Y_1, \ldots, Y_n)$ is

$$\begin{aligned} Mn_{\mathscr{X}}^{n}(f|\boldsymbol{\theta}) &= \textstyle\sum_{\mathbf{z} \in \mathscr{X}^N} f(\mathbf{z}) \prod_{x \in \mathscr{X}} \theta_x^{T_x(\mathbf{z})} = \sum_{\mathbf{m} \in \mathscr{N}_{\mathscr{X}}^{N}} \sum_{\mathbf{z} \in [\mathbf{m}]} f(\mathbf{z}) \prod_{x \in \mathscr{X}} \theta_x^{m_x} \\ &= \textstyle\sum_{\mathbf{m} \in \mathscr{N}_{\mathscr{X}}^{n}} MuHy_{\mathscr{X}}^{N}(f|\mathbf{m}) \nu(\mathbf{m}) \prod_{x \in \mathscr{X}} \theta_x^{m_x} =: CoMn_{\mathscr{X}}^{n}\big(MuHy_{\mathscr{X}}^{n}(f|\cdot)\big|\boldsymbol{\theta}\big). \end{aligned}$$

It follows from Theorem 2 that the multinomial distribution $Mn_{\mathscr{X}}^{n}(\cdot|\boldsymbol{\theta})$ is exchangeable, and that $CoMn_{\mathscr{X}}^{n}(\cdot|\boldsymbol{\theta})$ is the corresponding distribution for the corresponding count vectors $\mathbf{T}_{\mathscr{X}}^{n}(Y_1, \ldots, Y_n)$. Therefore the sequence of IID random variables $Y_1, \ldots, Y_n, \ldots$ is exchangeable.

For any gamble f on $\mathscr{X}^N$, and as a function of $\boldsymbol{\theta}$, $Mn_{\mathscr{X}}^{N}(f|\boldsymbol{\theta})$ is always a polynomial gamble on $\Sigma_{\mathscr{X}}$. Now, if $\underline{R}_{\mathscr{X}}$ is any coherent lower prevision on the linear space $\mathscr{V}(\Sigma_{\mathscr{X}})$ of all polynomial gambles on the $\mathscr{X}$-simplex, then one can see that the family of coherent lower previsions $\underline{P}_{\mathscr{X}}^{n}$, $n \geq 1$, defined by $\underline{P}_{\mathscr{X}}^{n}(f) = \underline{R}_{\mathscr{X}}\big(Mn_{\mathscr{X}}^{n}(f|\cdot)\big)$, for any $f \in \mathscr{L}(\mathscr{X}^n)$ is still exchangeable and time consistent, and the corresponding count distributions are given by $\underline{Q}_{\mathscr{X}}^{n}(f) = \underline{R}_{\mathscr{X}}\big(CoMn_{\mathscr{X}}^{n}(g|\cdot)\big)$ for any $g \in \mathscr{L}(\mathscr{N}_{\mathscr{X}}^{n})$. A converse result also holds:

Theorem 3 (Representation theorem for exchangeable sequences)**.** *Given a time consistent family of exchangeable coherent lower previsions $\underline{P}^n_{\mathscr{X}}$ on $\mathscr{L}(\mathscr{X}^n)$, $n \geq 1$, there is a* unique coherent lower prevision $\underline{R}_{\mathscr{X}}$ *on $\mathscr{V}(\Sigma_{\mathscr{X}})$ such that for all $n \geq 1$, all $f \in \mathscr{L}(\mathscr{X}^n)$ and all $g \in \mathscr{L}(\mathscr{N}^n_{\mathscr{X}})$ it holds that $\underline{P}^n_{\mathscr{X}}(f) = \underline{R}_{\mathscr{X}}(Mn^n_{\mathscr{X}}(f|\cdot))$ and $\underline{Q}^n_{\mathscr{X}}(g) = \underline{R}_{\mathscr{X}}\left(CoMn^n_{\mathscr{X}}(g|\cdot)\right)$.*

The crucial step in proving this consists in recognising that any representing $\underline{R}_{\mathscr{X}}$, if it exists, is only uniquely determined on $\mathscr{V}(\Sigma_{\mathscr{X}}) = \left\{CoMn^n_{\mathscr{X}}(g|\cdot) \colon n \geq 1, g \in \mathscr{L}(\mathscr{N}^n_{\mathscr{X}})\right\}$, each of whose elements is a polynomial function on the $\mathscr{X}$-simplex. So the basic idea is to *define* $\underline{R}_{\mathscr{X}}$ on $\mathscr{V}(\Sigma_{\mathscr{X}})$ as follows: *consider any element p of $\mathscr{V}(\Sigma_{\mathscr{X}})$. Then, by definition, there are some $n \geq 1$ and $g \in \mathscr{L}(\mathscr{N}^n_{\mathscr{X}})$ such that $p = CoMn^n_{\mathscr{X}}(g|\cdot)$. We then let $\underline{R}_{\mathscr{X}}(p) := \underline{Q}^n_{\mathscr{X}}(g)$.* This definition is consistent, and the functional $\underline{R}_{\mathscr{X}}$ thus defined on the linear space $\mathscr{V}_{\mathscr{X}}$ is indeed a coherent lower prevision.

Next, consider the sequence of so-called *frequency* random variables $\mathbf{F}_n := \mathbf{T}^n_{\mathscr{X}}(X_1, \dots, X_n)/n$ corresponding to an exchangeable sequence of random variables $X_1, \dots, X_n, \dots$, and assuming values in the $\mathscr{X}$-simplex $\Sigma_{\mathscr{X}}$. The distribution $\underline{P}_{\mathbf{F}_n}$, i.e., the coherent lower prevision on $\mathscr{L}(\Sigma_{\mathscr{X}})$ that models the available information about the values that $\mathbf{F}_n$ assumes in $\Sigma_{\mathscr{X}}$, is given by $\underline{P}_{\mathbf{F}_N}(h) = \underline{Q}^n_{\mathscr{X}}(h \circ \frac{1}{n}) = \underline{R}_{\mathscr{X}}\left(CoMn^n_{\mathscr{X}}(h \circ \frac{1}{n}|\cdot)\right)$ for any $h \in \mathscr{L}(\Sigma_{\mathscr{X}})$. So we find the following result, which provides an interpretation for the representation $\underline{R}_{\mathscr{X}}$, and which can be seen as another generalisation of de Finetti's Representation Theorem.

Theorem 4. *For all continuous gambles h on $\Sigma_{\mathscr{X}}$, $\lim_{n\to\infty} \underline{P}_{\mathbf{F}_n}(h) = \underline{R}_{\mathscr{X}}(h)$, and in this specific sense,* the sample frequencies $\mathbf{F}_n$ converge in distribution.

4 Extending local assessments to an exchangeable coherent lower prevision

In practice, a subject will usually make an assessment that N variables $X_1, \dots, X_N$ in a set $\mathscr{X}$ are exchangeable, in addition to specifying supremum acceptable buying prices $\underline{P}(f)$ for all gambles in some set of gambles $\mathscr{K} \subseteq \mathscr{L}(\mathscr{X}^N)$. *Can we turn these assessments into an exchangeable coherent lower prevision $\underline{P}^N_{\mathscr{X}}$ defined on all of $\mathscr{L}(\mathscr{X}^{\mathbb{N}})$, that is furthermore as small (least-committal, conservative) as possible?*

We start by symmetrising $\underline{P}$. On the symmetrised domain $\mathscr{K}_s := \{\pi f \colon f \in \mathscr{K}$ and $\pi \in \mathscr{P}_N\}$, define the lower prevision $\underline{P}_s$ by letting $\underline{P}_s(\pi f) := \underline{P}(f)$ for all $f \in \mathscr{K}$ and $\pi \in \mathscr{P}_N$. When $\mathscr{K}$ already contains gambles that are related to each other through (non-trivial) permutations, this may lead to conflicts, but then it makes good sense to define $\underline{P}_s$ on such gambles by taking the highest value that $\underline{P}$ takes on them. In this way, we end up with the point-wise smallest (most conservative) permutable (Walley, 1991) lower prevision $\underline{P}_s$ that dominates $\underline{P}$ on $\mathscr{K}$. Its natural extension $\underline{E}_s$ to all gambles is the point-wise smallest coherent extension of $\underline{P}_s$ to all gambles on $\mathscr{X}^N$, and can be determined efficiently (Walley, 1991).

Theorem 5. *If $\underline{P}_s$ avoids sure loss, then there are exchangeable coherent lower previsions on $\mathscr{L}(\mathscr{X}^N)$ that dominate $\underline{P}$ on its domain $\mathscr{K}$. The count distribution $\underline{Q}^N_{\mathscr{X}}$ of the point-wise smallest such exchangeable coherent lower prevision $\underline{P}^N_{\mathscr{X}}$ is given by $\underline{Q}^N_{\mathscr{X}}(h) = \underline{E}_s(\sum_{\mathbf{m}\in\mathscr{N}^N_{\mathscr{X}}} h(\mathbf{m}) I_{[\mathbf{m}]})$, for all gambles g on $\mathscr{N}^N_{\mathscr{X}}$.*

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