Desirability

A slightly different view of the basis of the theory of imprecise probabilities

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Outline

Desirable gambles

- Basic concepts
- Coherent sets of desirable gambles
- Assessments, avoiding partial loss & extensions
- Marginally desirable gambles & friends
- Conditioning sets of desirable gambles

Partial preference orders

Lower and upper previsions

- Moving from sets of desirable gambles to previsions
- Assessments, avoiding sure loss, natural extension & coherence
- Linear previsions & credal sets
- Conditioning lower previsions

Basic concepts: context & assumptions

Possibility space Ω e.g., outcomes experiment

Subject uncertain about outcome experiment

Goal model the subject's beliefs & use this model for reasoning

Gambles payoff depends on outcome,
bounded real-valued function on Ω,
set of gambles $\mathscr{L}_{\Omega}$

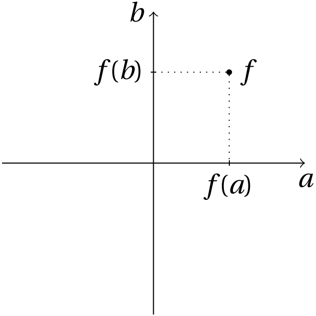
Utility linear and precise

Interpretation behavioral

Basic concepts: (desirable) gambles

$\Omega := \{a, b\}$

$f \in \mathscr{L}_\Omega$



Gamble f desirable when the subject accepts the transaction

(i) the experiment's outcome ω is determined

(ii) the subject's capital is changed by $f(\omega)$

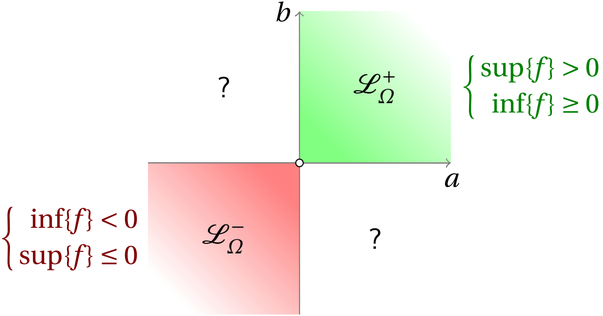
Subject's belief model set of desirable gambles

Rationality criteria for sets of desirable gambles: passive

To be *coherent*, a set of desirable gambles $\mathscr{R}$ must satisfy

Accepting partial gains: $\mathscr{L}_\Omega^+ \subseteq \mathscr{R}$,

Avoiding partial loss: $\mathscr{L}_\Omega^- \cap \mathscr{R} = \emptyset$.



The zero gamble is on the fence.

Rationality criteria for sets of desirable gambles: active

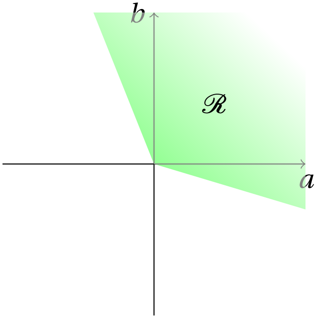
To be *coherent*, a set of desirable gambles $\mathscr{R}$ must also satisfy

Positive scaling:	$\lambda > 0, f \in \mathscr{R} \;\Rightarrow\; \lambda f \in \mathscr{R},$
Addition:	$f, g \in \mathscr{R} \;\Rightarrow\; f + g \in \mathscr{R}.$

A coherent set of desirable gambles $\mathscr{R}$ is a *convex cone* including $\mathscr{L}^+_\Omega$ and excluding $\mathscr{L}^-_\Omega$:

$$\lambda\mathscr{R} = \mathscr{R}$$

$$\mathscr{R} + \mathscr{R} = \mathscr{R}$$



Some people who have worked with cones

C.A.B. Smith 1961 implicitly, using an open cone of ‘exchange vectors’

P.M. Williams 1974 all main ideas present, uses the word ‘acceptable’
builds on B. de Finetti's work

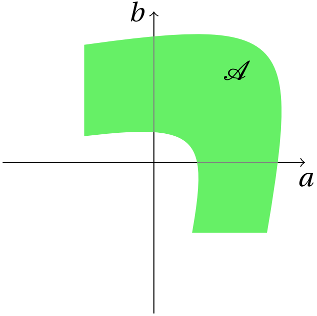
P. Walley 1991 focus on conceptual ease of updating,
builds on the above two

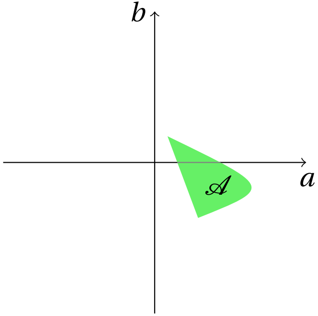
S. Moral 2000 epistemic irrelevance,
later work also with collaborators

G. de Cooman & E. Miranda 2007 symmetry

Assessments

An assessment is a set $\mathscr{A}$ of gambles judged desirable by the subject.

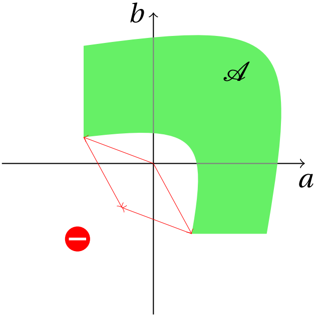




Conical hull operator coni all finite strictly positive linear combinations of elements of its argument set

Avoiding partial loss

An assessment $\mathscr{A}$ incurs partial loss iff $\operatorname{coni}(\mathscr{A}) \cap \mathscr{L}_{\Omega}^{-} \neq \emptyset$.
No coherent set of desirable gambles will encompass it then.

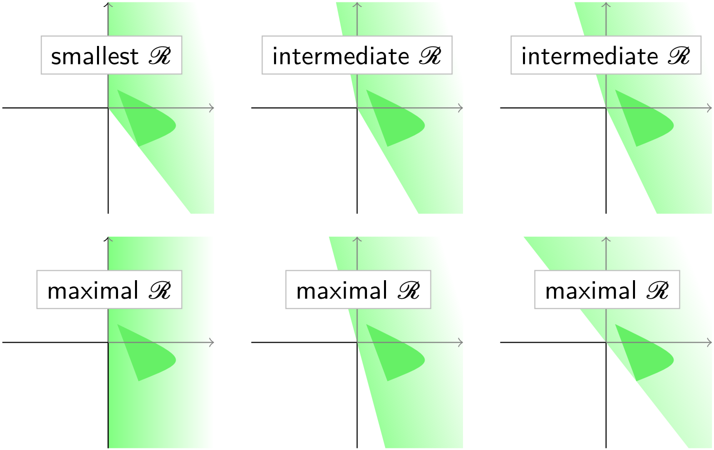


An assessment $\mathscr{A}$ *avoids partial loss* (and thus sure loss) iff

$$\operatorname{coni}(\mathscr{A}) \cap \mathscr{L}_{\Omega}^{-} = \emptyset$$

Least and maximally committal extensions

An assessment that avoids partial loss can have a continuum of encompassing coherent sets of desirable gambles $\mathscr{R}$

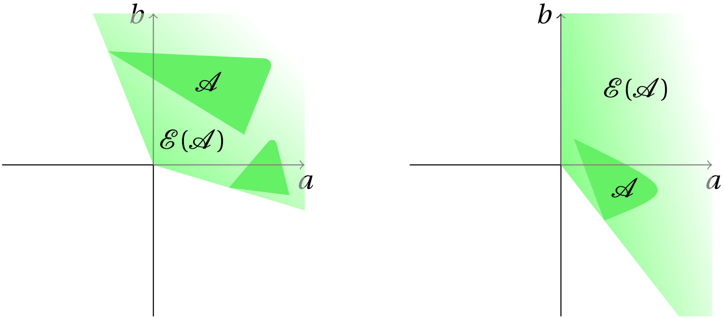


Maximal ones are not always (closed or open) half-spaces!

Natural extension

The least committal extension of an assessment $\mathscr{A}$ is called its *natural extension*

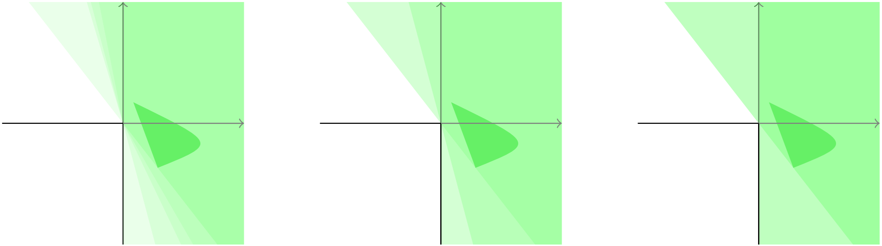
$$\mathscr{E}(\mathscr{A}) = \mathrm{coni}\big(\mathscr{A} \cup \mathscr{L}_{\Omega}^{+}\big).$$



The natural extension $\mathscr{E}(\mathscr{A})$ is coherent iff $\mathscr{A}$ avoids partial loss.

Natural extension as an intersection

The set of cones is closed under intersections.



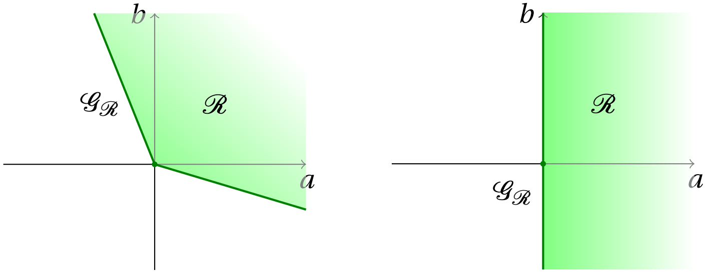
The natural extension of an assessment that avoids partial loss is the intersection of all

- coherent sets of desirable gambles that encompass it
- maximal elements of the coherent sets of desirable gambles that encompass it
- a minimal set of maximal elements of the coherent sets of desirable gambles that encompass it

Marginally desirable gambles

The set of marginally desirable gambles of $\mathscr{R}$ is

$$\mathscr{G}_{\mathscr{R}} := \{f - \sup\{\alpha \in \mathbb{R} \mid f - \alpha \in \mathscr{R}\} \mid f \in \mathscr{R}\}$$
$$= \mathrm{bd}(\mathscr{R}) \quad \text{when } \mathscr{R} \text{ coherent.}$$



Marginally desirable gambles & friends

The set of marginally desirable gambles of $\mathscr{R}$ is

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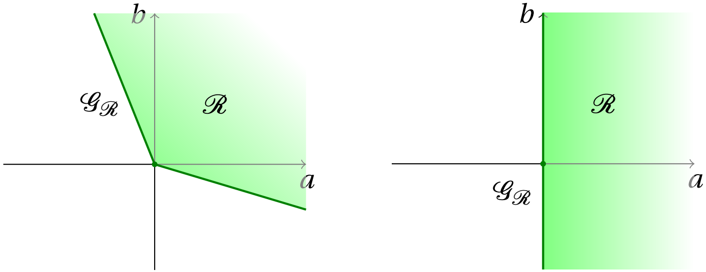


The set of almost desirable gambles is $\mathrm{cl}(\mathscr{R})$, the closure of $\mathscr{R}$

The set of strictly desirable gambles is $\mathrm{int}(\mathscr{R}) \cup \mathscr{L}_{\Omega}^{+}$,
the interior of $\mathscr{R}$ and gambles from $\mathscr{L}_{\Omega}^{+}$

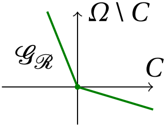
Updating a set of desirable gambles

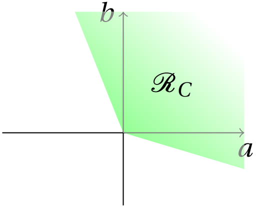
Indicator I_C of an event $C \subseteq \Omega$ is the gamble $I_C(\omega) := \begin{cases} 1, & \omega \in C, \\ 0, & \text{otherwise.} \end{cases}$

Updated or contingent set of desirable gambles of a set of desirable gambles $\mathscr{R}$ on an event C:

$$\mathscr{R}_C := \{f_C \mid f \in \mathscr{R} ; f = fI_C\} \subseteq \mathscr{L}_C$$

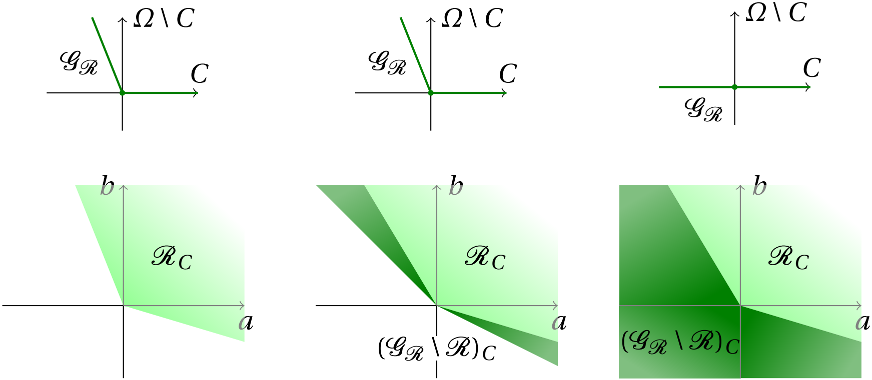
$\Omega \supset C = \{a, b\}$





Updating a coherent set of desirable gambles results in a coherent updated set of desirable gambles.

Updated sets of desirable gambles & marginally desirable gambles



Things left dangling

- Sets of desirable gambles relative to a linear subspace of $\mathscr{L}_\Omega$

- Transformations of Ω and sets of desirable gambles

- The issue of the zero gamble

- Combining different sets of desirable gambles

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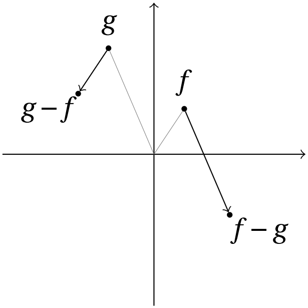
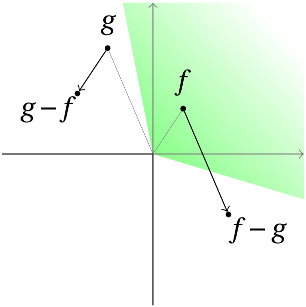
Difference of gambles



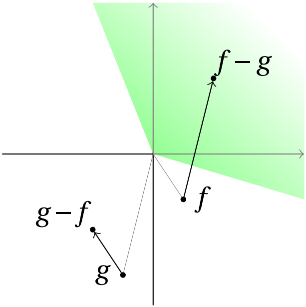
Comparing & ordering gambles

Partial preference order $\succeq$ associated to
a coherent set of desirable gambles $\mathscr{R}$:

$$f \succeq g \Leftrightarrow f - g \in \mathscr{R}$$

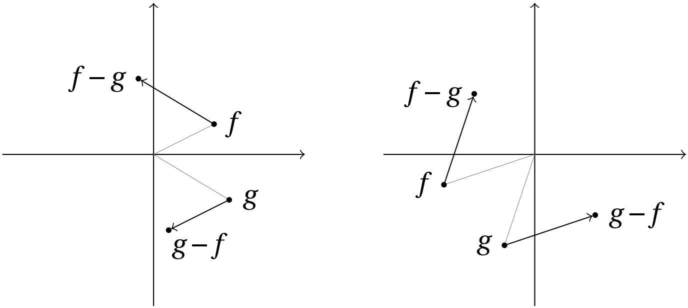


$f \not\succeq g$ and $g \not\succeq f$



$f \succeq g$ and $g \not\succeq f$

Difference of gambles



Comparing & ordering gambles

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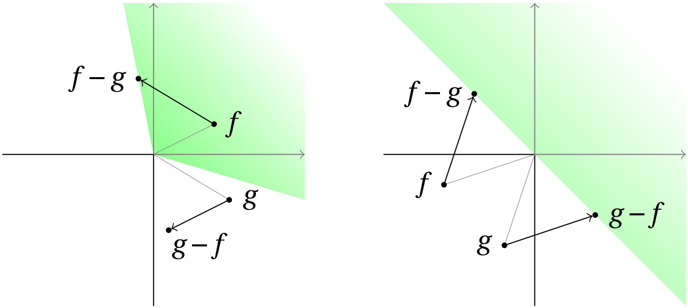


$f \overset{?}{\succeq} g$ and $g \not\succeq f$

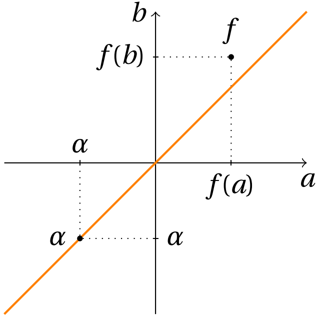
$f \overset{?}{\succeq} g$ and $g \overset{?}{\succeq} f$

Outline

Constant gambles

$f \in \mathscr{L}_\Omega$

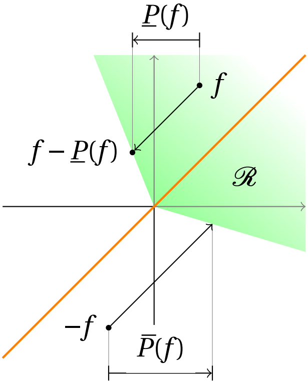
$\alpha \in \mathbb{R}$

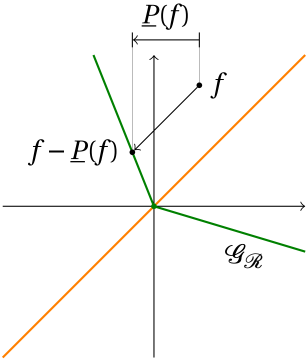


- The set of constant gambles can be linearly ordered (trivially so).
- What happens if we compare a gamble f with all constant gambles?
- Use constant gambles for buying and selling gambles.

From (marginally) desirable gambles to previsions

Lower and upper previsions $\underline{P}$ and $\bar{P}$ corresponding to a coherent set of desirble gambles $\mathscr{R}$:



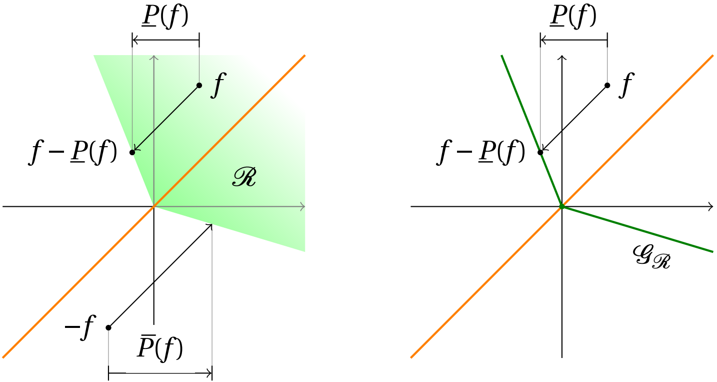


$$\underline{P}(f) := \sup\{\alpha \in \mathbb{R} \mid f \geq \alpha\} = \sup\{\alpha \in \mathbb{R} \mid f - \alpha \in \mathscr{R}\}$$

$$= \alpha \text{ such that } f - \alpha \in \mathscr{G}_{\mathscr{R}}$$

From (marginally) desirable gambles to previsions

Lower and upper previsions $\underline{P}$ and $\overline{P}$ corresponding to a coherent set of desirble gambles $\mathscr{R}$:

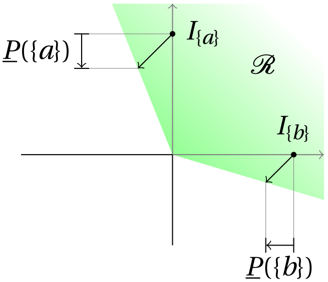


$$\begin{aligned}\overline{P}(f) := \inf\{\beta \in \mathbb{R} \mid \beta \geq f\} &= \inf\{\beta \in \mathbb{R} \mid \beta - f \in \mathscr{R}\}\\ &= -\underline{P}(-f)\end{aligned}$$

Lower and upper probabilities

Lower and upper probabilities of the event A:

$$\underline{P}(A) := \underline{P}(I_A) \qquad \overline{P}(A) := \overline{P}(I_A) = 1 - \underline{P}(\Omega \setminus A)$$



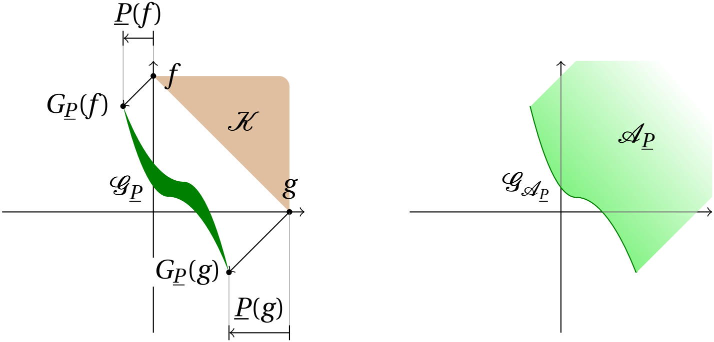
Lower and upper probabilities are less expressive than lower and upper previsions, which are less expressive than sets of desirable gambles.

From previsions to assessments

What is the assessment corresponding to a lower prevision $\underline{P}$ on $\mathcal{K} \subseteq \mathcal{L}_\Omega$?

The set of marginally desirable gambles of $\underline{P}$ is

$$\mathcal{G}_{\underline{P}} := \{G_{\underline{P}}(f) \mid f \in \mathcal{K}\}, \quad \text{with } G_{\underline{P}}(f) = f - \underline{P}(f).$$



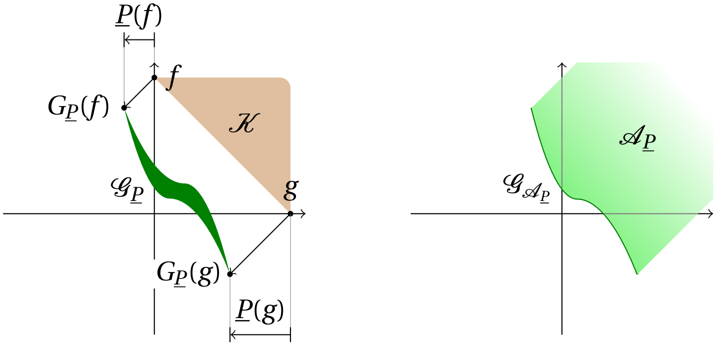
$$\mathcal{A}_{\underline{P}} := \mathcal{G}_{\underline{P}} + \mathbb{R}_{>0} = \{g + \alpha \mid g \in \mathcal{G}_{\underline{P}};\, \alpha > 0\}$$

Avoiding sure loss

A lower previsions $\underline{P}$ on $\mathcal{K} \subseteq \mathcal{L}_\Omega$ *avoids sure loss* iff

- $\mathscr{A}_{\underline{P}}$ avoids sure loss, or, as it is an open set, iff $\mathscr{A}_{\underline{P}}$ avoids partial loss

$$\operatorname{coni}(\mathscr{A}_{\underline{P}}) \cap \mathcal{L}_\Omega^- = \emptyset \quad \Leftrightarrow \quad \forall g \in \operatorname{coni}(\mathscr{G}_{\underline{P}}) : \sup\{g\} \geq 0$$

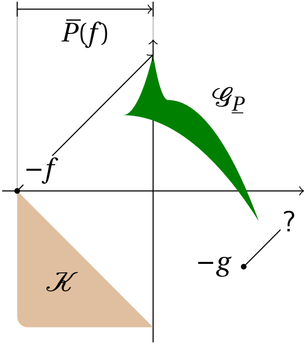


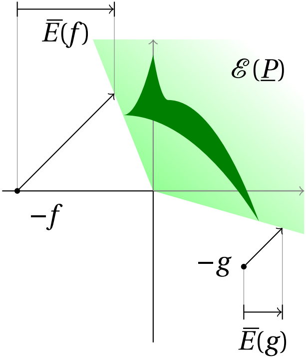
$$\mathscr{A}_{\underline{P}} := \mathscr{G}_{\underline{P}} + \mathbb{R}_{>0} = \{g + \alpha \mid g \in \mathscr{G}_{\underline{P}}; \alpha > 0\}$$

Natural extension of a lower prevision

The natural extension $\underline{E}$ of a lower previsions $\underline{P}$ on $\mathscr{K} \subseteq \mathscr{L}_\Omega$ is

$$\begin{aligned}\underline{E}(f) &:= \sup\{\alpha \in \mathbb{R} \mid f - \alpha \in \mathscr{E}(\underline{P})\}, \quad \text{with } \mathscr{E}(\underline{P}) := \mathscr{E}(\mathscr{A}_{\underline{P}}),\\ &= \sup\{\alpha \in \mathbb{R} \mid f - \alpha \geq g; g \in \operatorname{coni}(\mathscr{G}_{\underline{P}})\}.\end{aligned}$$





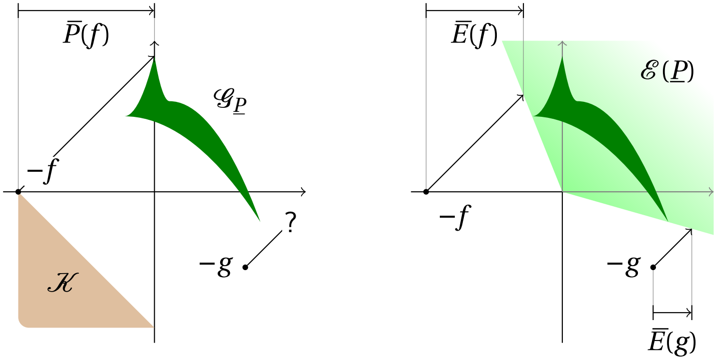
Coherence of a lower prevision

A lower previsions $\underline{P}$ on $\mathcal{K} \subseteq \mathcal{L}_\Omega$ is *coherent* iff

$$\forall f \in \mathcal{K} : \underline{P}(f) = \underline{E}(f)$$

$$\Leftrightarrow$$

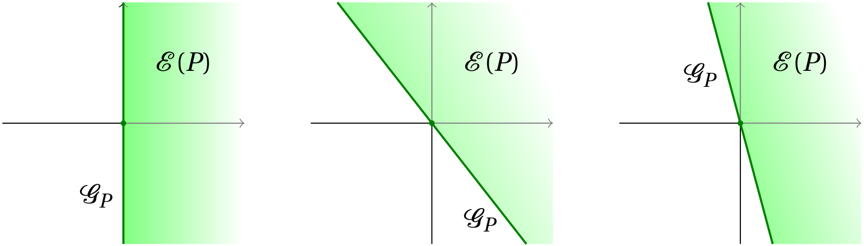
$$\forall f \in \mathcal{G}_{\underline{P}} : \forall g \in \mathrm{coni}(\mathcal{G}_{\underline{P}}) : \sup\{g - f\} \geq 0.$$



Linear previsions & hyperplanes

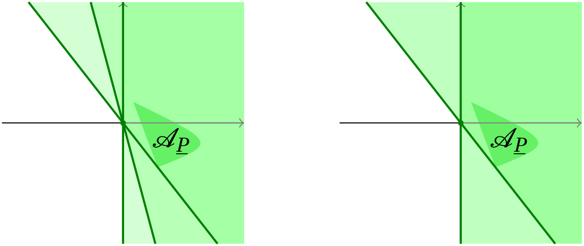
A linear prevision P corresponds to a set of marginally desirable gambles $\mathscr{G}_P$ that is a hyperplane:

$$P\big(G_P(f)\big) = P\big(f - P(f)\big) = 0 = P\big(P(f) - f\big) = P\big(G_P(-f)\big).$$



Maximally committal extensions & credal sets

The natual extension $\mathscr{E}(\underline{P})$ of a lower prevision $\underline{P}$ that avoids sure loss is the intersection of the set of maximally committal extensions.



The credal set $\mathscr{M}_{\underline{P}}$ is the closed convex set of linear previsions

$$\mathscr{M}_{\underline{P}} := \{P \text{ on } \mathscr{L}_{\Omega} \in \mid P \geq \underline{P}\}$$

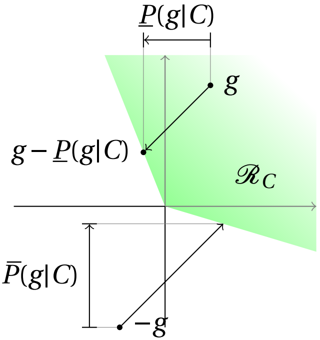
The natural extension $\underline{E}$ of $\underline{P}$ can then be written as

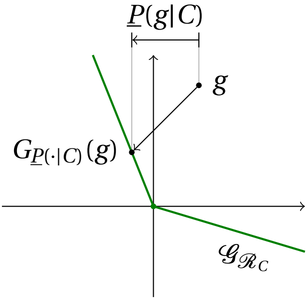
$$\underline{E}(f) = \min\{P \in \mathscr{M} \mid P(f)\} = \min\{P \in \operatorname{ext} \mathscr{M}_{\underline{P}} \mid P(f)\}.$$

Updating a coherent lower prevision

Updated or contingent lower prevision of a set of desirable gambles $\mathscr{R}$ or a coherent lower prevision $\underline{P}$ on $\mathscr{L}_\Omega$ (then $\mathscr{R} := \mathscr{E}(\underline{P})$):

$$\underline{P}(g|C) := \sup\{\alpha \in \mathbb{R} \mid g - \alpha \in \mathscr{R}_C\}$$
$$= \sup\{\alpha \in \mathbb{R} \mid (g - \alpha) I_C \in \mathscr{R}\}$$





The updated lower prevision $\underline{P}(\cdot|C)$ is a coherent lower prevision on $\mathscr{L}_C$.

Updating with natural and regular extension

Natural extension or the generalized Bayes's rule; alternative formulations:

$$\underline{P}(g|C) = \mu \in \mathbb{R} \text{ for which } \underline{P}\big((g-\mu)I_C\big) = 0$$
$$= \min\big\{P(gI_C)/P(C) \mid P \in \mathscr{M}_{\underline{P}}\big\}$$

whenever $\underline{P}(C) > 0$, otherwise $\underline{P}(\cdot|C)$ is vacuous.

Updating with natural and regular extension

Natural extension or the generalized Bayes's rule; alternative formulations:

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$$= \min\{P(gI_C)/P(C) \mid P \in \mathscr{M}_{\underline{P}}\}$$

whenever $\underline{P}(C) > 0$, otherwise $\underline{P}(\cdot|C)$ is vacuous.

Regular extension

$$\underline{R}(g|C) = \max\{\mu \in \mathbb{R} \mid \underline{P}\big((g-\mu)I_C) \geq 0\}$$
$$= \inf\{P(gI_C)/P(C) \mid P \in \mathscr{M}_{\underline{P}}; P(C) > 0\}$$

whenever $\overline{P}(C) > 0$, otherwise $\underline{R}(\cdot|C)$ is vacuous.

Updating with natural and regular extension

Natural extension or the generalized Bayes's rule; alternative formulations:

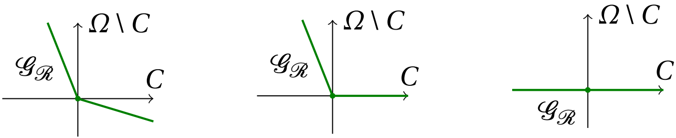
$$\underline{P}(g|C) = \mu \in \mathbb{R} \text{ for which } \underline{P}\big((g-\mu)I_C\big) = 0$$
$$= \min\big\{P(gI_C)/P(C) \,|\, P \in \mathscr{M}_{\underline{P}}\big\}$$

whenever $\underline{P}(C) > 0$, otherwise $\underline{P}(\cdot|C)$ is vacuous.

Regular extension

$$\underline{R}(g|C) = \max\big\{\mu \in \mathbb{R} \,|\, \underline{P}\big((g-\mu)I_C) \geq 0\big\}$$
$$= \inf\big\{P(gI_C)/P(C) \,|\, P \in \mathscr{M}_{\underline{P}}; P(C) > 0\big\}$$

whenever $\overline{P}(C) > 0$, otherwise $\underline{R}(\cdot|C)$ is vacuous.



$\underline{P}(C) > 0$ | $\overline{P}(C) > \underline{P}(C) = 0$ | $\overline{P}(C) = 0$

Updating on events of lower probability zero with natural and regular extension

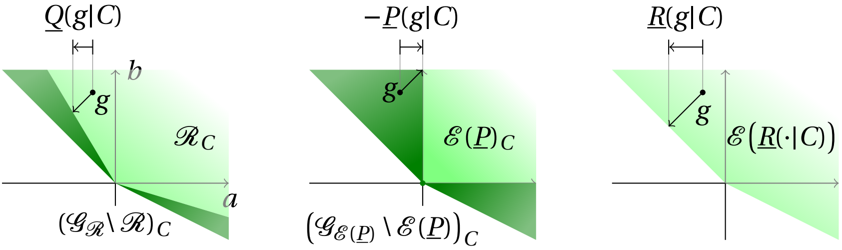
Consider a coherent set of desirable gambles $\mathscr{R}$

- $\underline{Q}(\cdot|C)$ is the lower prevision associated to $\mathscr{R}_C$
- $\underline{P}$ is the lower prevision associated to $\mathscr{R}$
  and $\overline{P}(C) > \underline{P}(C) = 0$, so $\underline{P}(\cdot|C)$ is vacuous
- $\underline{R}(\cdot|C)$ is the regular extension of $\underline{P}$

Updating on events of lower probability zero with natural and regular extension

Consider a coherent set of desirable gambles $\mathscr{R}$

- $\underline{Q}(\cdot|C)$ is the lower prevision associated to $\mathscr{R}_C$
- $\underline{P}$ is the lower prevision associated to $\mathscr{R}$ and $\overline{P}(C) > \underline{P}(C) = 0$, so $\underline{P}(\cdot|C)$ is vacuous
- $\underline{R}(\cdot|C)$ is the regular extension of $\underline{P}$

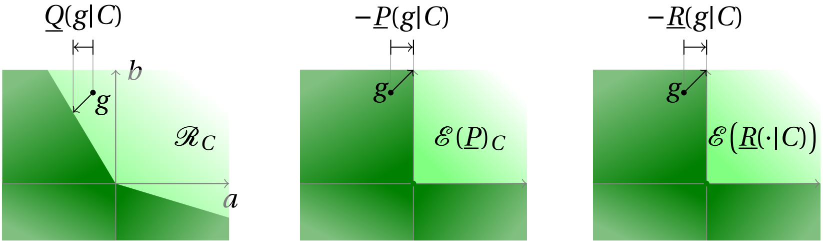


$$\inf\{g\} = \underline{P}(g|C) \leq \underline{Q}(g|C) \leq \underline{R}(g|C)$$

Updating on events of upper probability zero with natural and regular extension

Consider a coherent set of desirable gambles $\mathscr{R}$

- $\underline{Q}(\cdot|C)$ is the lower prevision associated to $\mathscr{R}_C$
- $\underline{P}$ is the lower prevision associated to $\mathscr{R}$
  and $\overline{P}(C) = \underline{P}(C) = 0$, so $\underline{P}(\cdot|C)$ is vacuous
- $\underline{R}(\cdot|C)$ is the regular extension of $\underline{P}$; it is also vacuous



$$\inf\{g\} = \underline{P}(g|C) = \underline{R}(g|C) \leq \underline{Q}(g|C)$$

Outline

Desirable gambles

- Basic concepts
- Coherent sets of desirable gambles
- Assessments, avoiding partial loss & extensions
- Marginally desirable gambles & friends
- Conditioning sets of desirable gambles

Partial preference orders

Lower and upper previsions

- Moving from sets of desirable gambles to previsions
- Assessments, avoiding sure loss, natural extension & coherence
- Linear previsions & credal sets
- Conditioning lower previsions

Advocating desirability

- A conceptually simple basis for imprecise-probability theory
- Intuitive and pedagogical due to its geometric nature
- Conditioning is elegant
- Holds promise for studying structural assessments

Combining updated sets of desirable gambles

- Given an assessent $\mathscr{A}_A$ for each event A in a set of events $\mathscr{A}$ of Ω
- These can be combined in an assessment on Ω:

$$\mathscr{A} := \bigcup_{A \in \mathscr{A}} \{gI_A \mid g \in \mathscr{A}_A\}$$

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- What about infinite partitions?
- $\mathscr{B}$-Conglomerability:

$$\forall B \in \mathscr{B} : f_B \in \mathscr{A}_B \quad \Rightarrow \quad f \in \mathscr{A}$$

('Full conglomerability' if it must hold for all partitions $\mathscr{B}$ of Ω.)

Conditional previsions & marginal gambles

Conditional lower prevision $\underline{P}(\cdot|\mathscr{B})$ for a partition $\mathscr{B}$ of Ω:

$$\underline{P}(f|\mathscr{B})(\omega) := \underline{P}(f|B) = \underline{P}(f_B|B) \text{ when } \omega \in B$$

It is a *piecewise constant gamble*, constant on $B \in \mathscr{B}$.

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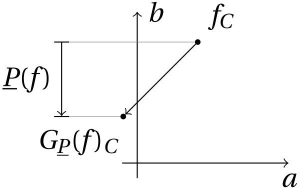

$\Omega := A \cup B$

$\mathscr{B} := \{A, B\}$

$C := \{a, b\}$

$a \in A$

$b \in B$

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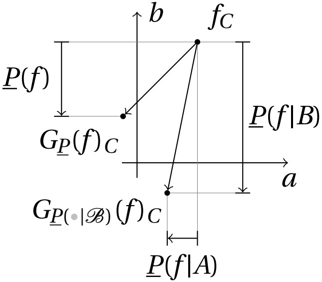

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Marginal gamble of a conditional prevision:

$$G_{\underline{P}(\cdot|\mathscr{B})}(f) = f - \underline{P}(f|\mathscr{B})$$

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