Taking lower probabilities to their extreme

Erik Quaeghebeur

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  Attach numbers/values to something.
- Be practical: make it meaningful/operational.
  Give maximum buying and minimum selling price to cost-functions/gambles.

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- A lower probability as a vector, e.g.,
  $\underline{P} = (\underline{P}_\emptyset, \underline{P}_{\{a\}}, \underline{P}_{\{b\}}, \underline{P}_{\{c\}}, \underline{P}_{\{a,b\}}, \underline{P}_{\{a,c\}}, \underline{P}_{\{b,c\}}, \underline{P}_\Omega)$.

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N.B.: All these criteria can be expressed as linear constraints on $\underline{P}$.

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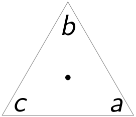

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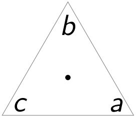
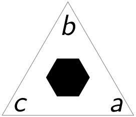


Coherent lower probabilities: As a convex subset of the $(|\Omega| - 1)$-dimensional unit simplex.



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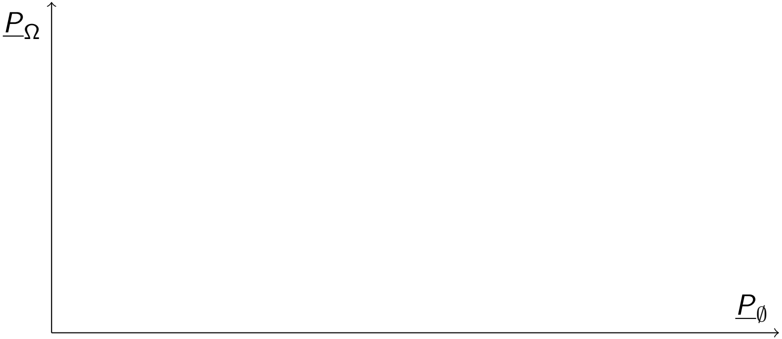

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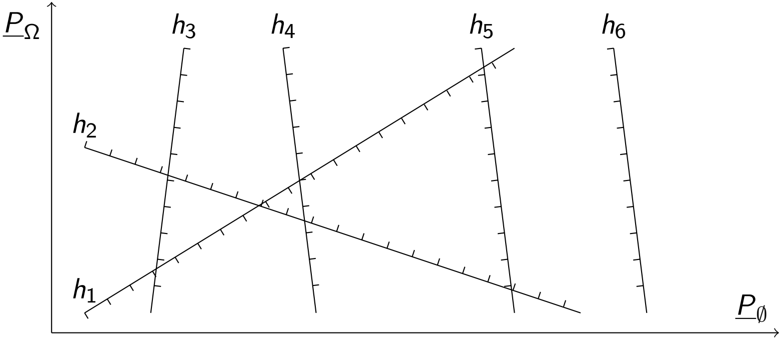

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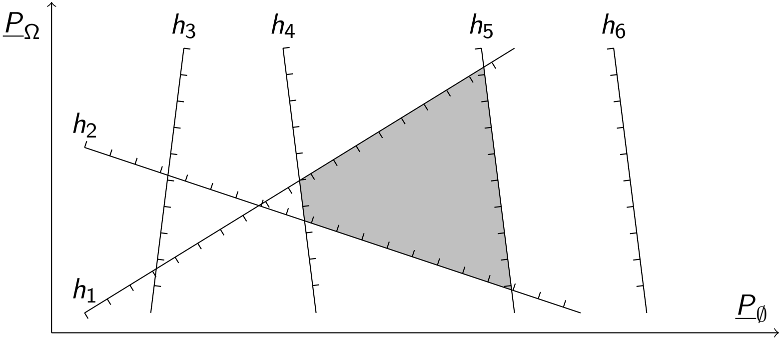

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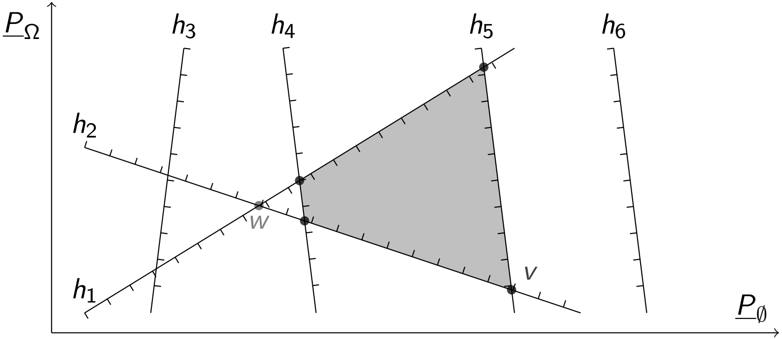

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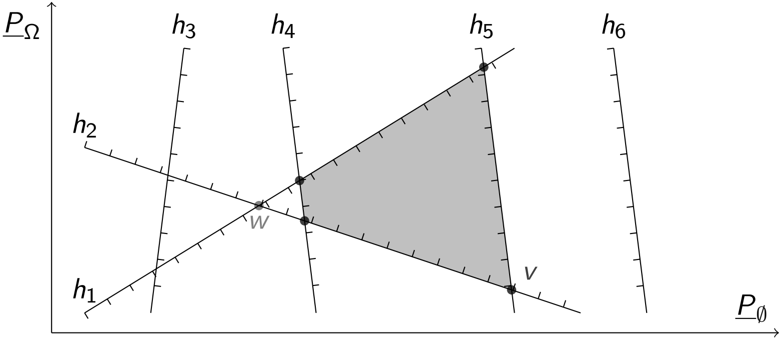

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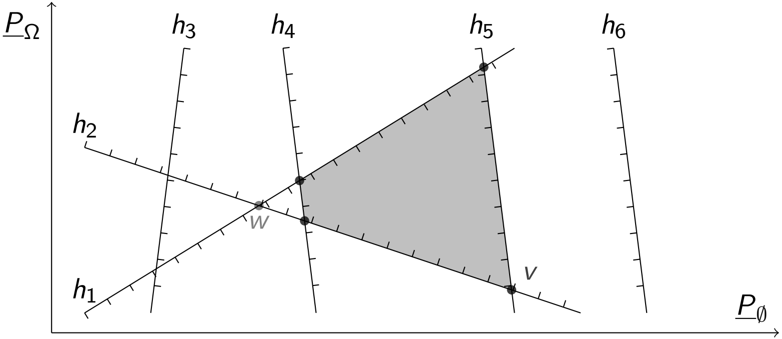

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Vertex enumeration From constraints to vertices: computationally heavy for large possibility spaces ($|\Omega| > 4$).

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  - Consider only linearly independent sets of B's or A and B's for which $\bigcup_B B = \Omega$ or $\bigcup_B B = A$.

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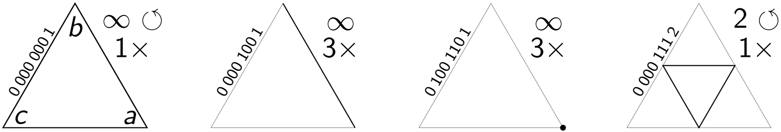

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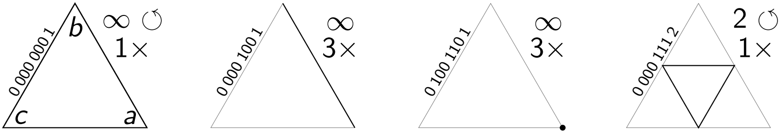

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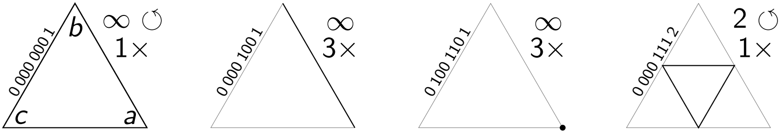

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- For $|\Omega| = 5$ there are at least 1 743 093.

Conclusions

- Lower probabilities can be used to model uncertainty.
- We assembled and successfully used
  a method to compute extreme lower probabilities.

- Questions:
  - More efficient (vertex enumeration) algorithms?
  - Direct calculation possible?
  - What about extreme lower previsions?