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Semi-global practical asymptotic stability and averaging

Andrew R. Teel[a], *, Joan Peuteman[b], Dirk Aeyels[b]

[a] *Electrical and Computer Engineering Department, University of California, Santa Barbara, CA 93106, USA*
[b] *SYSTeMS, Universiteit Gent, Technologiepark-Zwijnaarde, 9, 9052 Gent (Zwijnaarde), Belgium*

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Abstract

We prove a generalized Liapunov theorem which guarantees practical asymptotic stability. Based on this theorem, we show that if the averaged system $\dot{x} = f_{\mathrm{av}}(x)$ corresponding to $\dot{x} = f(x,t)$ is globally asymptotically stable then, starting from an arbitrarily large set of initial conditions, the trajectories of $\dot{x} = f(x, t/\varepsilon)$ converge uniformly to an arbitrarily small residual set around the origin when $\varepsilon > 0$ is taken sufficiently small. In other words, the origin is semi-globally practically asymptotically stable. © 1999 Elsevier Science B.V. All rights reserved.

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1. Introduction

The classical Liapunov approach to uniform asymptotic stability of the null solution of a dynamical system $\dot{x} = f(x,t)$ requires the existence of a positive definite, decrescent Liapunov function $V(x,t)$ whose derivative along the solutions of the system is negative definite. A weaker condition still guaranteeing uniform asymptotic stability has been established in [1,8]. Roughly stated, the origin is uniformly asymptotically stable under the condition that there exists a $d > 0$ and a sequence of times t_i such that for a positive-definite decrescent $V(x,t)$, $\forall x(t_i)$: $V(x(t_{i+1}), t_{i+1}) - V(x(t_i), t_i) \leqslant -\alpha_3(|x(t_i)|)$. Here, $\alpha_3(\cdot)$ is a strictly increasing continuous function on $\mathbb{R}^+$ which is zero at the origin, $t_{i+1} - t_i \leqslant \mathrm{d}\ \forall i \in \mathbf{N}$ with $t_i \to \infty$ as $i \to \infty$ and $x(t_{i+1})$ denotes the solution of $\dot{x} = f(x,t)$ evaluated at t_{i+1} with initial condition $x(t_i)$ at t_i. When there exists a positive $\lambda_{\max}$ and a positive $\lambda_{\min}$ such that $\forall t$, $\forall x$: $\lambda_{\min} x^{\mathrm{T}} x \leqslant V(x,t) \leqslant \lambda_{\max} x^{\mathrm{T}} x$, and there exists a positive $d > 0$ and a sequence of times t_i such that $V(x(t_{i+1}), t_{i+1}) - V(x(t_i), t_i) \leqslant -v|x(t_i)|^2$ with $v > 0$, $t_{i+1} - t_i \leqslant \mathrm{d}\ \forall i \in \mathbf{N}$ and $t_i \to \infty$ as $i \to \infty$ then the equilibrium point of $\dot{x} = f(x,t)$ is exponentially stable [2].

The present paper focuses on $(\sigma \to \rho)$-stability as defined in Section 2. We prove that $(\sigma \to \rho)$-stability is implied by the existence of a positive-definite decrescent Liapunov function $V(x,t)$, a positive $V_{\max}$, a positive $V_{\min}$, and a sequence of times t_i ($\exists d > 0$: $\forall i \in \mathbf{N}: t_{i+1} - t_i \leqslant d$) such that $V(x(t_{i+1}), t_{i+1}) - V(x(t_i), t_i) \leqslant -(t_{i+1} - t_i)\alpha_3(\inf_{s \in [t_i, t_{i+1}]} V(x(s), s))$ if $V(x(t_i), t_i) < V_{\max}$ and $\inf_{s \in [t_i, t_{i+1}]} V(x(s), s) \geqslant V_{\min}$.

These generalized Liapunov theorems are useful for establishing averaging results for time-varying systems. These averaging results are formulated in a Liapunov context, i.e., without using a state transformation as is typically done in averaging theory

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* Corresponding author.

E-mail addresses: teel@ece.ucsb.edu (A.R. Teel), Joan.Peuteman@rug.ac.be (J. Peuteman), Dirk.Aeyels@rug.ac.be (D. Aeyels)

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[3,5,9]. Relying on a state transformation, Hale [3] proves that for ε sufficiently small the periodic or almost periodic system $\dot{x} = f(x, t/\varepsilon, \varepsilon)$ has a unique periodic or almost periodic solution in a neighborhood of the equilibrium point 0 of the averaged system $\dot{x} = f_{\mathrm{av}}(x)$ when the real part of the eigenvalues of $(\partial f_{\mathrm{av}}/\partial x)(0)$ are different from zero. This unique solution is uniformly asymptotically stable when the linearization of the averaged system is exponentially stable [3]. When $f(0, t/\varepsilon, \varepsilon) = 0$ for all t and all $\varepsilon > 0$, this unique periodic or almost periodic solution is the null solution. In that case, it is proved by Khalil [5] that exponential stability of the averaged system implies exponential stability of the equilibrium point of $\dot{x} = f(x, t/\varepsilon, \varepsilon)$ for ε sufficiently small, without the assumption of periodicity or almost periodicity.

It is possible to prove this exponential averaging result [5] without using a state transformation. Indeed, the generalized Liapunov theorem for exponential stability of [2] is instrumental in proving that exponential stability of the averaged system implies exponential stability of the original time-varying system $\dot{x} = f(x, t/\varepsilon)$ for ε sufficiently small.

The generalized Liapunov theorems for uniform asymptotic stability of [1,8] allow to prove that asymptotic stability of the averaged system implies uniform asymptotic stability of the original time-varying system in case the system is homogeneous with positive order.

In the general case, however, asymptotic stability of the null solution of the averaged system does not imply uniform asymptotic stability of $\dot{x} = f(x, t/\varepsilon)$ with ε sufficiently small. But, by the generalized Liapunov theorem of the present paper, we are able to establish that global uniform asymptotic stability of the averaged system implies semi-global practical asymptotic stability of the origin of the original time-varying system $\dot{x} = f(x, t/\varepsilon)$ with ε sufficiently small; i.e., for all numbers σ and ρ with $\infty > \sigma > \rho > 0$, $\dot{x} = f(x, t/\varepsilon)$ is $(\sigma \to \rho)$-stable for ε sufficiently small. The main lines of the argument are as follows. First, the converse theorem of Liapunov for asymptotic stability of the averaged system guarantees the existence of a Liapunov function $V(x)$ whose derivatives along the flow of the averaged system is negative definite. Second, this $V(x)$ is used to establish $(\sigma \to \rho)$-stability of the origin of the original fast time-varying system. In general, the derivative of $V(x)$ along the flow of $\dot{x} = f(x, t/\varepsilon)$ may take positive as well as negative values which precludes the use of a classical theorem of Liapunov. For ε sufficiently small, we prove there exists a positive $V_{\max}$ and a positive $V_{\min}$, a sequence of times t_i ($\exists d > 0 : \forall i \in \mathbf{N} : t_{i+1} - t_i \leqslant d$) such that $V(x(t_{i+1}), t_{i+1}) - V(x(t_i), t_i) \leqslant -(t_{i+1} - t_i)\alpha_3(\inf_{s \in [t_i, t_{i+1}]} V(x(s), s))$ if $V(x(t_i), t_i) < V_{\max}$ and $\inf_{s \in [t_i, t_{i+1}]} V(x(s), s) \geqslant V_{\min}$. By Theorem 1, this implies $(\sigma \to \rho)$-stability.

Notice that it is possible to prove this averaging result without using a state transformation or a Lyapunov approach. In [7], the proof technique is based on the property that the solutions of $\dot{x} = f(x, t/\varepsilon)$ converge uniformly on compact time intervals to the solutions of the averaged system when ε approaches 0. Relying on this convergence property, it is proved in [7] that global uniform asymptotic stability of the averaged system implies semi-global practical asymptotic stability of the original time-varying system.

In [4], Hapaev studies stability, asymptotic stability and instability of systems arising as $\dot{x} = f_1(x, t) + \varepsilon f_2(x, t)$, where $\dot{x} = f_1(x, t)$ is uniformly stable, by a Liapunov approach related to our approach. His results are used to establish averaging results for multi-frequency systems $\dot{x} = \varepsilon X(x, q)$ $\dot{q} = \omega(x) + \varepsilon \Phi(x, q)$ where X and Φ are periodic in q. Asymptotic stability of the averaged of a multi-frequency system implies that $\forall \rho > 0$, $\exists \delta > 0$ and $\exists \varepsilon^* > 0$ such that for $\varepsilon < \varepsilon^*$, $|x(t_0)| < \delta$ implies that $|x(t)| < \rho$ $\forall t \geqslant t_0$. The averaging result of Theorem 2 of the present paper is not focused on multi-frequency systems — it is not restricted to periodic systems or systems with a specific structure.

2. Generalized Liapunov theorem

We first define the notion of $(\sigma \to \rho)$-stability.

Definition 1. $(\sigma \to \rho)$-*stability*. Given $\sigma > \rho \geqslant 0$, the origin of the system $\dot{x} = f(x, t)$ is said to be $(\sigma \to \rho)$-stable if

1. for each $\varepsilon > \rho$ there exists $\delta(\varepsilon) > 0$ such that for all $t_0 \geqslant 0$

$$|x(t_0)| \leqslant \delta(\varepsilon) \;\Rightarrow\; |x(t)| \leqslant \varepsilon \quad \forall t \geqslant t_0,$$

2. for each $r \in (0, \sigma)$ there exists a finite $\nu(r) > 0$ such that

$$|x(t_0)| \leqslant r \;\Rightarrow\; |x(t)| \leqslant \nu(r) \quad \forall t \geqslant t_0,$$

3. for each $r \in (0, \sigma)$ and each $\varepsilon > \rho$ there exists a finite $T(r, \varepsilon) > 0$ such that for all $t_0 \geqslant 0$

$$|x(t_0)| \leqslant r \;\Rightarrow\; |x(t)| \leqslant \varepsilon \quad \forall t \geqslant t_0 + T(r, \varepsilon).$$

Remark 2.1. In the case where $\sigma=\infty$ and $\rho=0$ this is the standard definition of uniform global asymptotic stability, equivalent to the existence of a class-$\mathcal{KL}$ function[1] β such that $|x(t)|\leqslant\beta(|x(t_0)|,t-t_0)$ for any initial state $x(t_0)$.

Next we write down the stability theorem that we will use to prove our main result. Here, $(\sigma\to\rho)$-stability is guaranteed by the existence of a Liapunov function that is decreasing when examined at certain sampling times.

Theorem 1. *Consider the system* $\dot{x}=f(x,t)$ *where* $f(x,t)$ *is locally Lipschitz in* x *uniformly in* t. *Let* $\sigma>\rho\geqslant 0$ *be given. If there exist a function* $V(x,t)$, *two class-*$\mathcal{K}_\infty$ *functions* α_1 *and* α_2, *a class-*$\mathcal{K}$ *function* α_3, *nonnegative real numbers* $V_{\min}, V_{\max}\in\mathbb{R}_{\geqslant 0}\cup\{\infty\}$, *a positive real number* d *and, for each pair* (x_0,t_0) *with* $|x_0|<\sigma$, *an increasing sequence* $\{t_i\}_{i=1}^{\infty}$ *with* $t_i\to\infty$ as $i\to\infty$ *such that*

1. $t_{i+1}-t_i\leqslant d$ *for all* $i=0,\ldots,\infty$,
2.
   2.1. $\alpha_1(|x|)\leqslant V(x,t)\leqslant\alpha_2(|x|)$
   2.2. *the trajectory* $x(t)$ *starting from* (x_0,t_0) *satisfies, for* $i\geqslant 1$: *if* $V(x(t_i),t_i)<V_{\max}$ *then* $x(t)$ *is defined on* $[t_i,t_{i+1}]$ *and if, moreover,*

$$\inf_{s\in[t_i,t_{i+1}]} V(x(s),s)\geqslant V_{\min}, \tag{1}$$

then

$$V(x(t_{i+1}),t_{i+1})-V(x(t_i),t_i) \leqslant -(t_{i+1}-t_i)\alpha_3\left(\inf_{s\in[t_i,t_{i+1}]} V(x(s),s)\right), \tag{2}$$

3. $|x(t_0)|<\sigma\Rightarrow V(x(t_1),t_1)<V_{\max}$,
4. *there exists* $\mu\geqslant 0$ *such that,* $\mu<V_{\max}$ *and for each pair* (x_0,t_0), *the resulting trajectory* $x(t)$ *satisfies*

$$|x(\tau)|\leqslant\alpha_1^{-1}(V_{\min}) \;\Rightarrow\; |x(t)|\leqslant\alpha_2^{-1}(\mu),$$
$$|x(\tau)|\leqslant\alpha_1^{-1}(\mu) \;\Rightarrow\; |x(t)|\leqslant\rho,$$

for all $t\in[\tau,\tau+d]$.

5. *for each* $r\in(0,\sigma)$ *there exists a finite* $v_1(r)>0$ *and a finite* $\bar{V}(r)\in[\mu,V_{\max})$ such that $|x(t_0)|\leqslant r$ *implies that* $x(t)$ *is defined and* $|x(t)|\leqslant v_1(r)$ *when* $t\in[t_0,t_1]$ *and* $V(x(t_1),t_1)\leqslant\bar{V}(r)$. *For* $i\geqslant 1$ *if* $V(x(t_i),t_i)\leqslant\bar{V}(r)$ *there exists a finite* $v_2(r)>0$ *such that* $|x(t)|\leqslant v_2(r)$ *when* $t\in[t_i,t_{i+1}]$.

then the origin of $\dot{x}=f(x,t)$ *is* $(\sigma\to\rho)$*-stable.*

Remark 2.2. When the origin is an equilibrium point, i.e., $f(0,t)=0$ $\forall t$, then for each $\rho>0$ there exists $\mu>0$ and, in turn, $V_{\min}>0$ such that item 4 is satisfied. Again for the case when the origin is an equilibrium point, if $\rho=V_{\min}=0$ we can take $\mu=0$.

Proof. *Stability*: Let $\varepsilon>\rho$. Assume, without loss of generality, that $\varepsilon<\sigma$. From item 4 and continuity of solutions uniform in the starting time and a compact set of initial conditions (due to the uniform local Lipschitz assumption on $f(x,t)$) there exists $\delta_v(\varepsilon)\in(\mu,\alpha_1(\sigma))$ such that

$$|x(\tau)|\leqslant\alpha_1^{-1}(\delta_v(\varepsilon)) \;\Rightarrow\; |x(t)|\leqslant\varepsilon, \tag{3}$$

for all $t\in[\tau,\tau+d]$. Using $\alpha_1(|x|)\leqslant V(x,t)$ we have

$$V(x(\tau),\tau)\leqslant\delta_v(\varepsilon) \;\Rightarrow\; |x(t)|\leqslant\varepsilon \tag{4}$$

for all $t\in[\tau,\tau+d]$. Also from item 4 there exists $\delta_{v2}(\varepsilon)\in(V_{\min},\delta_v(\varepsilon))$ such that

$$|x(\tau)|\leqslant\alpha_1^{-1}(\delta_{v2}(\varepsilon)) \;\Rightarrow\; |x(t)|\leqslant\alpha_2^{-1}(\delta_v(\varepsilon)), \tag{5}$$

for all $t\in[\tau,\tau+d]$. Define $\delta(\varepsilon):=\alpha_1^{-1}(\delta_{v2}(\varepsilon))$ and note that $\delta(\varepsilon)<\sigma$. Fix (x_0,t_0) with $|x_0|\leqslant\delta(\varepsilon)$. This fixes the sequence t_i. Then using (5) we have $V(x(t_1),t_1)\leqslant\delta_v(\varepsilon)$. From item 3, we have $V(x(t_1),t_1)<V_{\max}$. Therefore, using (3) we have $|x(t)|\leqslant\varepsilon$ for all $t\in[t_0,t_2]$. If $\inf_{s\in[t_1,t_2]}V(x(s),s)\geqslant V_{\min}$ then, using item 2b we have $V(x(t_2),t_2)\leqslant\delta_v(\varepsilon)-(t_2-t_1)\alpha_3(V_{\min})\leqslant\delta_v(\varepsilon)$ and $V(x(t_2),t_2)<V_{\max}-(t_2-t_1)\alpha_3(V_{\min})<V_{\max}$. We can repeat the argument. Otherwise, there exists $s\in[t_1,t_2]$ such that $|x(s)|\leqslant\alpha_1^{-1}(V_{\min})$. But, again from item 4, we then have $|x(t_2)|\leqslant\alpha_2^{-1}(\mu)$. Finally, using $V(x,t)\leqslant\alpha_2(|x|)$ we have $V(x(t_2),t_2)\leqslant\mu<\delta_v(\varepsilon)$ and $V(x(t_2),t_2)\leqslant\mu<V_{\max}$. So, in either case the argument can be repeated for all t_i and the first part of the $(\sigma\to\rho)$-stability definition is satisfied.

Boundedness: Let $r\in(0,\sigma)$. Fix (x_0,t_0) such that $|x_0|\leqslant r$. From item 5, we have $|x(t)|\leqslant v_1(r)$ when $t\in[t_0,t_1]$ and $V(x(t_1),t_1)\leqslant\bar{V}(r)<V_{\max}$. From item 5, $|x(t)|\leqslant v_2(r)$ when $t\in[t_1,t_2]$. If $\inf_{s\in[t_1,t_2]}V(x(s),s)\geqslant V_{\min}$ then using 2b we have $V(x(t_2),t_2)\leqslant\bar{V}(r)<V_{\max}$. We can repeat the argument. Otherwise, there exists $s\in[t_1,t_2]$ such that

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[1] A continuous function $\alpha:[0,a)\to[0,\infty)$ is said to be a class-$\mathcal{K}$ function if it is strictly increasing and $\alpha(0)=0$. It is said to be a class-$\mathcal{K}_\infty$ function if $a=\infty$ and $\alpha(r)\to\infty$ as $r\to\infty$. A continuous function $\sigma:[0,\infty)\to[0,\infty)$ is said to be a class-$\mathcal{L}$ function if it is strictly decreasing and $\sigma(s)\to 0$ as $s\to\infty$. A continuous function $\beta:[0,a)\times[0,\infty)$ is said to be a class-$\mathcal{KL}$ function if for each fixed s the mapping $\beta(r,s)$ is a class-$\mathcal{K}$ function with respect to r, and for each fixed r the mapping $\beta(r,s)$ is decreasing with respect to s and $\beta(r,s)\to 0$ as $s\to\infty$.

$|x(s)|\leqslant\alpha_1^{-1}(V_{\min})$. But from item 4, we then have $|x(t_2)|\leqslant\alpha_2^{-1}(\mu)$. Finally, using $V(x,t)\leqslant\alpha_2(|x|)$ we have $V(x(t_2),t_2)\leqslant\mu\leqslant\bar{V}(r)<V_{\max}$. In either case the argument can be repeated for all t_i and the second part of the $(\sigma\to\rho)$-stability definition is satisfied by taking $v(r)=\max\{v_1(r),v_2(r)\}$.

Convergence: Let $r\in(0,\sigma)$. Let $\varepsilon>\rho$ and assume without loss of generality, that $\varepsilon<\sigma$. Use the $\delta_v(\varepsilon)$ and $\delta_{v2}(\varepsilon)$ from the stability proof. Define $T(r,\varepsilon)>0$ as

$$T(r,\varepsilon):=\max\left\{0,\frac{\bar{V}(r)-\delta_{v2}(\varepsilon)}{\alpha_3(\delta_{v2}(\varepsilon))}\right\}+2d. \tag{6}$$

Fix (x_0,t_0) such that $|x_0|\leqslant r$. This fixes the sequence t_i and the maximal interval of definition $\mathscr{I}$. Suppose we have

$$\min_{s\in[t_0,t_0+T(r,\varepsilon)]\cap\mathscr{I}}|x(s)|\leqslant\alpha_1^{-1}(\delta_{v2}(\varepsilon)). \tag{7}$$

Then, from (5) and the proof of stability we conclude that $\mathscr{I}=[t_0,\infty)$ and

$$|x(t)|\leqslant\varepsilon,\quad \forall t\geqslant t_0+T(r,\varepsilon). \tag{8}$$

If (7) does not hold then we have

$$\inf_{s\in[t_0,t_0+T(r,\varepsilon)]\cap\mathscr{I}}V(x(s),s)\geqslant\delta_{v2}(\varepsilon)>V_{\min}. \tag{9}$$

We also have $V(x(t_1),t_1)\leqslant\bar{V}(r)<V_{\max}$. It follows from item 2b that $V(x(t_k),t_k)\leqslant\bar{V}(r)<V_{\max}$ for all $t_k\in[t_0,t_0+T(r,\varepsilon)]\cap\mathscr{I}$ and that $[t_0,t_0+T(r,\varepsilon)]\subset\mathscr{I}$. By repeated application of (2) we have, for the largest i such that $t_{i+1}\leqslant t_0+T(r,\varepsilon)$, $V(x(t_{i+1}),t_{i+1})-V(x(t_1),t_1)\leqslant-(t_{i+1}-t_1)\alpha_3(\delta_{v2}(\varepsilon))$. From item 1 we must have $t_{i+1}-t_1\geqslant T(r,\varepsilon)-2d$ so, using (6) we have

$$V(x(t_{i+1}),t_{i+1})\leqslant\delta_{v2}(\varepsilon)<\delta_v(\varepsilon). \tag{10}$$

Finally, from the stability part of the proof this implies $|x(t)|\leqslant\varepsilon$ for all $t\geqslant t_0+T(r,\varepsilon)$. □

3. Main result

In the present section, we prove that global asymptotic stability of the averaged system $\dot{x}=f_{\text{av}}(x)$ implies semi-global practical asymptotic stability of the original system, i.e., the trajectories of $\dot{x}=f(x,t/\varepsilon)$ converge uniformly to an arbitrary small residual set around the origin from an arbitrary large set of initial conditions as $\varepsilon>0$ approaches zero. We first define what is meant by a time-varying function having an average.

Definition 2. The function $f(x,t)$ is said to have an average $f_{\text{av}}(x)$ if there exists a class-$\mathscr{KL}$ function β, a class-$\mathscr{L}$ function σ and a $T^*>0$ such that for each $T\geqslant T^*$ and for all $t\geqslant0$

$$\left\|f_{\text{av}}(x)-\frac{1}{T}\int_t^{t+T}f(x,\tau)\,\mathrm{d}\tau\right\|\leqslant\beta(|x|,T)+\sigma(T). \tag{11}$$

Definition 3. *Semi-global practical asymptotic stability.* The origin of the system $\dot{x}=f(x,t/\varepsilon)$ with $\varepsilon>0$ is said to be semi-globally practically asymptotically stable if for all numbers σ and ρ with $\infty>\sigma>\rho>0$, there exists ε^* such that for $\varepsilon\in(0,\varepsilon^*)$, the corresponding system $\dot{x}=f(x,t/\varepsilon)$ is $(\sigma\to\rho)$-stable.

Theorem 2. *Consider the system*

$$\dot{x}=f(x,t/\varepsilon),\quad \varepsilon>0, \tag{12}$$

where $f(x,t)$ is locally Lipschitz in x uniformly in t and $f(0,t)$ is uniformly bounded (not necessarily zero). Suppose $f(x,t)$ has average $f_{\text{av}}(x)$ and that the origin of $\dot{x}=f_{\text{av}}(x)$ is globally asymptotically stable. $f_{\text{av}}(x)$ is locally Lipschitz in x. Then, the origin of (12) *is semi-globally practically asymptotically stable.*

Proof. By standard converse Liapunov theorems ([10] p. 440, [6]), there exist class-$\mathscr{K}_\infty$ functions α_1, α_2 and α_3 and a smooth function $V:\mathbb{R}^n\to\mathbb{R}_{\geqslant0}$ such that

$$\alpha_1(|x|)\leqslant V(x)\leqslant\alpha_2(|x|), \tag{13}$$

and

$$\frac{\partial V}{\partial x}(x)f_{\text{av}}(x)\leqslant-\alpha_3(V(x)). \tag{14}$$

Let σ and ρ be given and let L be a Lipschitz constant for $f(x,t)$, $f_{\text{av}}(x)$ and $\partial V/\partial x$ over a ball of radius $\alpha_1^{-1}\circ\alpha_2(2\sigma)$. Let $c>0$ satisfy

$$|f(0,t)|\leqslant c,\quad \forall t\geqslant0. \tag{15}$$

Then we can write, for all x_1 and x_2 belonging to a ball of radius $\alpha_1^{-1}\circ\alpha_2(2\sigma)$,

$$\left|\frac{\partial V}{\partial x}(x_1)f(x_1,t)-\frac{\partial V}{\partial x}(x_2)f(x_2,t)\right| \leqslant(L|x_1|+L|x_2|+c)L|x_1-x_2|, \tag{16}$$

and similarly with $f(\cdot,t)$ replaced by $f_{\text{av}}(\cdot)$. Also note that, as long as trajectories remain in a ball of radius

$\alpha_1^{-1}\circ\alpha_2(2\sigma)$ we have with the Gronwall lemma and

$$\begin{aligned}|x(t)-x(\tau)| \leqslant{}& |t-\tau|(c+L|x(\tau)|)\\ &+L\left|\int_\tau^t |x(s)-x(\tau)|\,\mathrm{d}s\right| \qquad (17)\end{aligned}$$

that

$$|x(t)-x(\tau)|\leqslant \left[|x(\tau)|+\frac{c}{L}\right][\exp(L|t-\tau|)-1]. \qquad (18)$$

So we can choose $d>0$ sufficiently small such that the solutions of (12) satisfy

$$|x(\tau)|\leqslant\alpha_1^{-1}\circ\alpha_2(\sigma) \Rightarrow \ |x(t)|\leqslant\alpha_1^{-1}\circ\alpha_2(2\sigma) \qquad (19)$$

for all $t\in[\tau,\tau+d]$ and that, for the given ρ, there exists $V_{\min}>0$ and $\mu>0$ such that $\mu<\alpha_2(\sigma)$ and item 4 of Theorem 1 is satisfied. Let this fix $V_{\min}>0$ and pick $V_{\max}=\alpha_2(\sigma)$.

Fix (x_0,t_0) with $|x_0|<\sigma$ and define the sequence $t_i=t_0+(i-1)\varepsilon T$. Here $\varepsilon>0$ and $T>0$ are taken such that $\varepsilon T\leqslant d$. This implies that $t_{i+1}-t_i=\varepsilon T\leqslant d$ for all $i=1,\dots,\infty$ and $t_1=t_0$. Suppose $V(x(t_i))<\alpha_2(\sigma)=V_{\max}$. This implies that $|x(t_i)|<\alpha_1^{-1}\circ\alpha_2(\sigma)$ which implies thanks to (19) that $x(t)$ is defined on $[t_i,t_{i+1}]$. In what follows, we use the shorthand $\bar V_i=\min_{s\in[t_i,t_{i+1}]}V(x(s))$. Integrating the derivative of $V(x(t))$ along trajectories we obtain that $V(x(t_{i+1}))-V(x(t_i))$ is less or equal than

$$\begin{aligned}&-\varepsilon T\alpha_3(\bar V_i)+\int_{t_i}^{t_{i+1}}\frac{\partial V}{\partial x}(x(t))[f(x(t),t/\varepsilon)\\ &\quad -f_{\mathrm{av}}(x(t))]\,\mathrm{d}t. \qquad (20)\end{aligned}$$

$V(x(t_{i+1}))-V(x(t_i))$ is less or equal than

$$\begin{aligned}&-\varepsilon T\alpha_3(\bar V_i)+\int_{t_i}^{t_{i+1}}\frac{\partial V}{\partial x}(x(t_i))[f(x(t_i),t/\varepsilon)\\ &\quad -f_{\mathrm{av}}(x(t_i))]\,\mathrm{d}t+2\int_{t_i}^{t_{i+1}}(L|x(t_i)|+Lq(|x(t_i)|)\\ &\quad +c)(L|x(t_i)|+c)(\mathrm{e}^{L(t-t_i)}-1)\,\mathrm{d}t. \qquad (21)\end{aligned}$$

For the last term we have used (16) and (18) such that $|x(t)|\leqslant q(|x(t_i)|)$ with the definition

$$q(r):=\mathrm{e}^{Ld}r+\frac{c}{L}(\mathrm{e}^{Ld}-1). \qquad (22)$$

If we choose $T\geqslant T^*$, we obtain that $V(x(t_{i+1}))-V(x(t_i))$ is less or equal than

$$\begin{aligned}&-\varepsilon T\alpha_3(\bar V_i)+\varepsilon TL|x(t_i)|(\beta(|x(t_i)|,T)+\sigma(T))\\ &\quad +2\varepsilon T(L|x(t_i)|+Lq(|x(t_i)|)+c)\\ &\quad (L|x(t_i)|+c)\bar q(\varepsilon T). \qquad (23)\end{aligned}$$

Here

$$\bar q(\varepsilon T):=\frac{\mathrm{e}^{L\varepsilon T}-(1+L\varepsilon T)}{L\varepsilon T}. \qquad (24)$$

By using (18) and (22), $|x(t_i)|\leqslant q(|x(t)|)\leqslant q(\alpha_1^{-1}(V(x(t),t)))$ for all $t\in[t_i,t_{i+1}]$ and in particular the one giving $V(x(t),t)=\bar V_i$. Therefore, $|x(t_i)|\leqslant q(\alpha_1^{-1}(\bar V_i))$ such that we obtain that $V(x(t_{i+1}))-V(x(t_i))$ is less or equal than

$$\begin{aligned}&-\varepsilon T\alpha_3(\bar V_i)+\varepsilon TLq(\alpha_1^{-1}(\bar V_i))(\beta(q(\alpha_1^{-1}(\bar V_i)),T)\\ &\quad +\sigma(T))+2\bar q(\varepsilon T)\varepsilon T(Lq(\alpha_1^{-1}(\bar V_i))\\ &\quad +Lq(q(\alpha_1^{-1}(\bar V_i)))+c)(Lq(\alpha_1^{-1}(\bar V_i))+c) \qquad (25)\end{aligned}$$

when $T\geqslant T^*$. Since β is a class-$\mathscr{KL}$ function, σ is a class-$\mathscr{L}$ function and α_3 is a class-$\mathscr{K}_\infty$ function, it is possible to choose $T\geqslant T^*$ sufficiently large and $\eta>0$ sufficiently small such that, for all $V\in[V_{\min},V_{\max}]$,

$$\begin{aligned}&-0.5\alpha_3(V)+Lq(\alpha_1^{-1}(V))(\beta(q(\alpha_1^{-1}(V)),T)+\sigma(T))\\ &\quad +2\eta(Lq(\alpha_1^{-1}(V))+Lq(q(\alpha_1^{-1}(V)))+c)\\ &\qquad (Lq(\alpha_1^{-1}(V))+c)\leqslant 0. \qquad (26)\end{aligned}$$

With this choice of T and η, take $\varepsilon>0$ sufficiently small such that not only $\varepsilon T\leqslant d$, but also $\bar q(\varepsilon T)\leqslant\eta$. When

$$\inf_{s\in[t_i,t_{i+1}]}V(x(s))\geqslant V_{\min}, \qquad (27)$$

we conclude from this calculation and (26) that $V(x(t_{i+1}))<\alpha_2(\sigma)$ and

$$\begin{aligned}&V(x(t_{i+1}))-V(x(t_i))\\ &\quad\leqslant -0.5(t_{i+1}-t_i)\alpha_3\left(\inf_{s\in[t_i,t_{i+1}]}V(x(s))\right).\end{aligned}$$

Since $t_0=t_1$, item 5 is satisfied. By taking for each $r\in(0,\sigma)$ $v_1(r)=r$ and $\bar V(r)=\max\{\alpha_2(r),\mu\}<V_{\max}=\alpha_2(\sigma)$. For $i\geqslant 1$, if $V(x(t_i),t_i)\leqslant\bar V(r)$ then $|x(t_i)|\leqslant\max\{\alpha_1^{-1}\circ\alpha_2(r),\alpha_1^{-1}(\mu)\}<\alpha_1^{-1}\circ\alpha_2(\sigma)$. Therefore, using (19) it is clear that item 5 is satisfied since $|x(t)|\leqslant\alpha_1^{-1}\circ\alpha_2(2\sigma)=:v_2(r)$ when $t\in[t_i,t_{i+1}]$.

The result then follows by applying Theorem 1 with the given σ and ρ. □

Remark 3.1. Note that when the origin of the averaged system is locally exponentially stable so that we can take $\alpha_1(s)\propto s^2$, $\alpha_2(s)\propto s^2$, $\alpha_3(s)\propto s$, when $f(0,t)=0$ so that $c=0$, when $\beta(|x|,T)\propto|x|$ with slope inversely proportional to T and when $\sigma(T)=0$ for all $T>0$ then we can pick T and η in (26) so that (26) holds even when $V_{\min}=0$. Thus, we recover well-known local averaging results when the origin of the averaged system is locally exponentially stable [2].

Another interesting local result comes about when the system is homogeneous with a positive order [8]. The Lipschitz constant approaches zero as the state is restricted to a smaller and smaller neighborhood of the origin. In this case, the averaging result holds even when ε is not small.

4. Conclusions

The paper establishes a generalized Liapunov theorem which is useful in proving that global asymptotic stability of the averaged system implies semi-global practical asymptotic stability of the original system.

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