On the number of samples necessary to achieve observability

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In [1] we have shown that almost all dynamical systems are observable with respect to an almost arbitrary sample program consisting of $2n+1$ samples (n is the dimension of the differentiable manifold supporting the dynamical system). In this paper we construct a dynamical system which is unobservable with respect to any sample program consisting of $2n$ samples. Small perturbations of the dynamics do not destroy the non-observability. This shows that the results obtained in [1] are the best ones possible in general.

Keywords: Observability, Sampled data systems, Differential geometric methods.

Let X be a smooth paracompact n-dimensional manifold. Let ξ belong to $\mathcal{X}^r(X)$, $r \geqslant 1$, the space of all C^r-vectorfields defined on X. Let h belong to $\mathcal{C}^r(X,\mathbb{R})$, $r \geqslant 1$, the space of all C^r-functions mapping into $\mathbb{R}$. Both $\mathcal{X}^r(X)$ and $\mathcal{C}^r(X,\mathbb{R})$ are endowed with the Whitney C^r-topology. Let P be a sample program, i.e. a finite set of different points t_i belonging to $[0,T]$, $T \in \mathbb{R}$. A system (ξ,h) is P-observable if and only if for each (x,y) with $x \neq y$, there exists a t_i belonging to P such that $h \circ \phi_{t_i}(x) \neq h \circ \phi_{t_i}(y)$. Here $\phi: \mathbb{R} \times X \to X$ denotes the flow associated with the vectorfield ξ under consideration.

In a forthcoming paper [1] we state a number of results related to the genericity aspect of observability. Typically, there exists an open and dense set $\Omega \subset [0,T]^{2n+1}$ (endowed with the subspace topology, derived from $\mathbb{R}^{2n+1}$), such that for each $P \in \Omega$, the set of P-observable pairs (ξ,h) is residual in $\mathcal{X}^r(X) \times \mathcal{C}^r(X,\mathbb{R})$. In other words, for almost any sample program consisting of $2n+1$ samples, observability is a generic property. In the above paper we have also constructed an example showing that the theorems of [1] are no longer valid in general if one restricts the sampling procedure to $2n$ samples. The example consists of a smooth vectorfield defined on S^1 and an output function $h: S^1 \to \mathbb{R}$ such that there are two sample times for which the system is not observable.

In addition the non-observability of the system is stable under small perturbations in the vectorfield, the output function, and the sample times. In other words, there exists an open subset Ω' of $[0,T]^2$, and an open set $\mathcal{Q}$ of $\mathcal{X}^r(S^1) \times \mathcal{C}^r(S^1,\mathbb{R})$, containing the system under consideration, such that for any sample program $P' \in \Omega'$ systems belonging to $\mathcal{Q}$ are not observable. Of course, there are no reasons a priori for rejecting observability of the present system with respect to sample programs in the complement of Ω'. Actually, the latter remark arose in a discussion with René Boel and has led to the writing of this paper.

We now describe the content of the paper. Choose an almost arbitrary dynamic system living on an n-dimensional manifold. We know from [1] that this system is P-observable for *almost all* programs $P \in [0,T]^{2n+1}$. One might conjecture that almost all dynamical systems are observable with respect to sample programs belonging to $[0,T]^k$, $k < 2n+1$, if one is willing to consider only *well chosen* sample programs in $[0,T]^k$. In this paper an example is designed to counter this supposed trade-off. An open set of systems living on an n-dimensional manifold is constructed. It is shown that all the systems belonging to this subset are not observable with respect to *any* sample program consisting of $2n$ samples no matter how cleverly chosen. Therefore the results obtained in [1] are the best ones possible in general.

Proposition. *For each integer $n \geqslant 1$, there exists an n-dimensional manifold X and a dynamical system (ξ^*, h^*) living on the manifold such that the system is unobservable for* all *sample programs consisting of $2n$ sample times. In addition, unobservability is* stable *with respect to small perturbations in the vectorfield and the output mapping.*

Proof. The idea behind the proof is—at least in the one-dimensional case—to construct a vectorfield $\xi^* \in \mathcal{X}^r(S^1)$ and an output mapping $h^* \in$

$\mathcal{C}^r(S^1,\mathbb{R})$ such that

$$h^* \circ \phi^*_{t_0}, h^* \circ \phi^*_{t_1} : S^1 \to \mathbb{R}^2$$

twists the circle into a figure eight, no matter how t_0 and t_1 are chosen (with the natural restriction $t_1 > t_0 \geqslant 0$). This implies that there are two points of S^1 which map into the crossing point of the figure eight. It is intuitively clear that this crossing cannot be resolved by perturbing the system. The two points are therefore indistinguishable causing the non-observability of the system in a stable way.

We now proceed to formalize this idea. First consider the case $n=1$. Let X be S^1. Let S^1 be realized in the (x,y)-plane as the circle with radius equal to 1 and center in (0,1). Consider a Morse–Smale vectorfield ξ^* on S^1 which has two equilibrium points, viz. a source in the north pole (0,2) and a sink in the south pole (0,0). Let ϕ^*: $\mathbb{R} \times S^1 \to S^1$ be the flow associated with ξ^*. For the definition of the output mapping, let S^1 be realized as $\mathbb{R}/2\pi$ and let h^* be induced on $\mathbb{R}/2\pi$ from the sin function. Identify the 'Euclidean circle S^1' with $\mathbb{R}/2\pi$ by wrapping $\mathbb{R}/2\pi$ around S^1 such that the north pole coincides with $2\pi\mathbb{Z}$. Formally the function h^* is the sin function defined on S^1 with zero value in the north pole and the south pole.

Let $[0,T]$ be a time interval available for observation, $T>0$. We will show that for *any two* sample times $t_0, t_1 \in [0,T]$ the mapping $(h^* \circ \phi^*_{t_0}, h^* \circ \phi^*_{t_1}) : S^1 \to \mathbb{R}^2$ is not injective. Furthermore, the noninjectiveness persists under small perturbations of (ξ^*, h^*) in the space $\mathcal{X}^r(S^1) \times \mathcal{C}^r(S^1,\mathbb{R})$.

First notice that without loss of generality one of the sample times can be taken equal to zero.

Let $t_1 \in (0,T]$. Consider the mapping

$$h^*, h^* \circ \phi^*_{t_1} : S^1 \to \mathbb{R}^2. \qquad (\S)$$

Both NP (north pole) and SP (south pole) map into $(0,0) \in \mathbb{R}^2$. We will show that this mapping has a normal crossing [2] at $(0,0) \in \mathbb{R}^2$.

Let v_{NP} and v_{SP} be unit vectors—with respect to the Euclidean metric in $\mathbb{R}^2$—belonging to the tangent plane of S^1 respectively in NP an SP and pointing in the clockwise direction of S^1.

Then

$$\mathrm{d}h^*(\mathrm{NP}) \cdot v_{\mathrm{NP}} = 1,$$

$$\mathrm{d}h^*(\mathrm{NP}) \cdot \mathrm{D}\phi^*_{t_1}(\mathrm{NP}) \cdot v_{\mathrm{NP}} = a$$

and

$$\mathrm{d}h^*(\mathrm{SP}) \cdot v_{\mathrm{SP}} = -1,$$

$$\mathrm{d}h^*(\mathrm{SP}) \cdot \mathrm{D}\phi^*_{t_1}(\mathrm{SP}) \cdot v_{\mathrm{SP}} = -b$$

with $a, b \in \mathbb{R}$ and positive. Arrange the vectorfield ξ^* such that the linearizations at NP and SP have eigenvalues unequal in modulus. Then a and b are different. The evaluation mapping

$$\Delta\big((\mathcal{X}^r(S^1) \times \mathcal{C}^r(S^1,\mathbb{R})) \times (\mathcal{X}^r(S^1) \times \mathcal{C}^r(S^1,\mathbb{R}))\big)$$
$$\times S^1 \times S^1 \to \mathbb{R}^2 \times \mathbb{R}^2,$$
$$\xi, h, \xi, h, x, y \mapsto \big(h(x), h \circ \phi_{t_1}(x)\big),\ \big(h(y), h \circ \phi_{t_1}(y)\big)$$

is a C^r-mapping. Here Δ denotes the diagonal.

The representation mapping corresponding to (ξ^*, h^*, ξ^*, h^*) maps (NP,SP) to $(0,0) \times (0,0)$. The above calculation shows that this representation mapping is transversal to the diagonal $\Delta(\mathbb{R}^2 \times \mathbb{R}^2)$, thereby proving normality of the intersection at $(0,0) \in \mathbb{R}^2$ of the mapping (§). All conditions are satisfied to apply the isotopy theorem [3] to the above evaluation mapping. The isotopy theorem implies that for a neighborhood of (ξ^*, h^*) in the space $\mathcal{X}^r(S^1) \times \mathcal{C}^r(S^1,\mathbb{R})$ the normal crossing persists. This ends the proof of the proposition in case $\dim X = 1$.

The method of proof can be rather directly extended to the higher dimensional case. Let X be the n-dimensional torus $S^1 \times \cdots \times S^1$. On each component define a Morse–Smale vectorfield with a source in NP and a sink in SP. The vectorfield on the n-dimensional torus is the cartesian product of the vectorfields on the components. The output mapping

$$h^* : S^1 \times \cdots \times S^2 \to \mathbb{R}$$

is defined by

$$(x_1, \ldots, x_n) \mapsto \sin x_1 + \cdots + \sin x_n$$

where sin represents the function induced on S^1 by the sin function defined on $\mathbb{R}$. Identify $S^1 \times \cdots \times S^1$ with $\mathbb{R}^n/\mathbb{Z}^n$ by identifying componentwise as done earlier. Then the mapping

$$h^*, h^* \circ \phi^*_{t_1}, \ldots, h^* \circ \phi^*_{t_{2n-1}} : S^1 \times \cdots \times S^1 \to \mathbb{R}^{2n}$$

maps $(\mathrm{NP}_1, \ldots, \mathrm{NP}_n)$ and $(\mathrm{SP}_1, \ldots, \mathrm{SP}_n)$ into $(0,\ldots,0)$. The proof of normal intersection at $(0,\ldots,0)$ is developed along the same lines as in the one-dimensional case.

Remark. Instead of invoking the isotopy theorem to show that small perturbations of (ξ^*, h^*) do not destroy the non-observability, we can also proceed as follows. First notice that, for each $t_1 > 0$, the mapping $h^*, h^* \circ \phi^*_{t_1} : S^1 \to \mathbb{R}^2$ is an immersion. We know from [2] that an immersion with normal crossings belonging to $\mathcal{C}^r(S^1, \mathbb{R}^2)$ is stable (for the definition of stability of a function in a space of functions, one is referred to [2]). Stability of $h^*, h^* \circ \phi^*_{t_1}$ in $\mathcal{C}^r(S^1, \mathbb{R}^2)$ implies among other things that there is an open neighborhood of $h^*, h^* \circ \phi^*_{t_1}$ in $\mathcal{C}^r(S^1, \mathbb{R}^2)$ having the same number of selfintersections. Since the mapping

$$\mathscr{X}^r(S^1) \times \mathcal{C}^r(S^1, \mathbb{R}) \to \mathcal{C}^r(S^1, \mathbb{R}^2),$$
$$\xi, h \mapsto (h, h \circ \phi_{t_1})$$

is continuous, we have shown that in a neighborhood of the system (ξ^*, h^*) the mappings $(h, h \circ \phi_{t_1})$ have selfintersections. This ends the proof.

References

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