

## AVERAGING RESULTS AND THE STUDY OF UNIFORM ASYMPTOTIC STABILITY OF HOMOGENEOUS DIFFERENTIAL EQUATIONS THAT ARE NOT FAST TIME-VARYING\*

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**Abstract.** Within the Liapunov framework, a sufficient condition for uniform asymptotic stability of ordinary differential equations is proposed. Unlike with classical Liapunov theory, the time derivative of the  $V$ -function, taken along solutions of the system, may have positive and negative values. It is shown that the proposed condition is useful for the study of uniform asymptotic stability of homogeneous systems with order  $\tau > 0$ . In particular, it is established that asymptotic stability of the averaged homogeneous system implies local uniform asymptotic stability of the original time-varying homogeneous system. This shows that averaging techniques play a prominent role in the study of homogeneous—not necessarily fast time-varying—systems.

**Key words.** nonlinear systems, homogeneous systems, averaging, asymptotic stability

**AMS subject classifications.** 34, 34D, 34D20, 93D20

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**1. Introduction.** The classical Liapunov approach to uniform asymptotic stability of the null solution of a dynamical system  $\dot{x}(t) = f(x, t)$  requires the existence of a positive definite, decrescent Liapunov function  $V(x, t)$  whose derivative along the solutions of the system is negative definite. When this derivative is negative semidefinite, stability rather than asymptotic stability follows; in case the differential equation is autonomous, the Barbashin–Krasovskii theorem or LaSalle’s invariance principle may be helpful in proving asymptotic stability. Extensions to periodic differential equations are possible. For nonperiodic systems, Narendra and Annaswamy [12] show that with  $\dot{V}(x, t) \leq 0$  uniform asymptotic stability can be proven if there exists a  $T \in \mathbb{R}^+$  such that  $\forall t : V(x(t+T), t+T) - V(x(t), t) \leq -\gamma(\|x(t)\|) < 0$ , where  $\gamma(\cdot)$  is a strictly increasing continuous function on  $\mathbb{R}^+$  which is zero at the origin and where  $x(t+T)$  is the solution of the system at  $t+T$  with initial condition  $x(t)$  at  $t$ . Weaker conditions than the Narendra–Annaswamy conditions also leading to asymptotic stability have recently been obtained [1, 2, 3, 4]. The present paper and [3, 4] have been inspired by the result of Narendra–Annaswamy. We claim that in the asymptotic stability theorem of Narendra–Annaswamy, *the negative semidefiniteness condition on the time-derivative of the  $V$ -function can be dispensed with*: the null solution of a differential equation is uniformly asymptotically stable under the condition that for a positive definite decrescent  $V(x, t)$ ,  $\exists T > 0$ , and a strictly increasing sequence of times  $t_k^*$  such that  $V(x(t_{k+1}^*), t_{k+1}^*) - V(x(t_k^*), t_k^*) \leq -\gamma(\|x(t_k^*)\|)$  with  $\gamma(\cdot) : \mathbb{R}^+ \rightarrow \mathbb{R}^+$  continuous, strictly increasing and passing through the origin and  $t_{k+1}^* - t_k^* \leq T \forall k \in \mathbb{Z}$  and  $t_k^* \rightarrow \infty$  as  $k \rightarrow \infty$  and  $t_k^* \rightarrow -\infty$  as  $k \rightarrow -\infty$ . Compared to [12],  $\dot{V}(x, t) \leq 0$  is no longer required. Compared to [3], the condition on  $V$  needs to be satisfied only for a sequence  $t_k^*$ , not for all  $t$ . Unlike [4], this paper is focused upon uniform asymptotic stability, not exponential stability.

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In section 2 of this paper, the classical theorem of Liapunov for discrete-time nonlinear systems is recalled. In section 3, we propose a sufficient condition which guarantees uniform asymptotic stability of a continuous-time nonlinear system. In section 4 and section 5, a proposition and a theorem are stated concerning uniform asymptotic stability of time-varying homogeneous systems  $\dot{x}(t) = f(x, \alpha t)$  ( $\forall \alpha \in \mathbb{R}_0^+$ ) with order  $\tau > 0$ . It is shown that asymptotic stability of the averaged (*time-invariant* homogeneous) system guarantees local uniform asymptotic stability of the original *time-varying* homogeneous system. For stability results on time-invariant homogeneous systems, the reader is referred to [7, 9]. An important—perhaps surprising—feature of our result is that it is valid *independent of*  $\alpha$ . This shows that averaging techniques play a prominent role in the study of time-varying—not necessarily fast time-varying—homogeneous systems of order  $\tau > 0$ .

As illustrated in section 6 and studied in [11], it is worthwhile mentioning that the averaging results for homogeneous systems with order  $\tau = 0$  are different in the sense that they are valid only for  $\alpha$  sufficiently large.

The region of attraction of the homogeneous system  $\dot{x}(t) = f(x, \alpha t)$  ( $\alpha \in \mathbb{R}_0^+$ ) with order  $\tau > 0$  depends on  $\alpha$ . This region of attraction increases when  $\alpha$  increases and grows unbounded as  $\alpha$  goes to infinity.

**2. Theorem of Liapunov for discrete-time systems.** Consider the time-varying discrete-time system

$$(1) \quad x_{k+1} = g(x_k, k)$$

with  $g : W_d \times \mathbb{Z} \rightarrow \mathbb{R}^n$ ,  $W_d$  open,  $W_d \subset \mathbb{R}^n$ . Let  $g(0, k) = 0 \forall k \in \mathbb{Z}$  and  $0 \in W_d$ .

**PROPOSITION 1.** *Consider a function  $V : U_d \times \mathbb{Z} \rightarrow \mathbb{R}$ , with  $U_d$  an open neighborhood of 0. We assume the following.*

*Condition 1.*  $V(x, k)$  is positive definite and decrescent; i.e.,  $V(0, k) = 0 \forall k$  and  $\forall x \in U_d : \alpha_V(\|x\|) \leq V(x, k) \leq \beta_V(\|x\|)$ . The functions  $\alpha_V(\cdot) : \mathbb{R}^+ \rightarrow \mathbb{R}^+$  and  $\beta_V(\cdot) : \mathbb{R}^+ \rightarrow \mathbb{R}^+$  are strictly increasing continuous functions passing through the origin.

*Condition 2.* There exists a function  $\gamma(\cdot) : \mathbb{R}^+ \rightarrow \mathbb{R}^+$ , which is continuous, strictly increasing, and passing through the origin, and an open set  $U'_d \subset U_d \cap W_d$ , which contains the origin such that  $\forall k \in \mathbb{Z} \forall x_k \in U'_d \setminus \{0\}$ :

$$(2) \quad V(x_{k+1}, k+1) - V(x_k, k) \leq -\gamma(\|x_k\|) < 0,$$

where  $x_{k+1} = g(x_k, k)$ .

*Then the equilibrium point  $x = 0$  of (1) is locally uniformly asymptotically stable.*

*Proof.* The present proposition is the classical theorem of Liapunov for uniform asymptotic stability for discrete-time nonlinear time-varying systems. For the proof of this proposition, the reader is referred to [5, 8].  $\square$

*Remark 1.* When the closed ball with radius  $\nu$  centered at 0,  $\overline{B}_\nu(0) \subset U'_d$  and  $x_{k_0} \in B_{\nu'}(0)$  with  $\nu' = \beta_V^{-1}(\alpha_V(\nu))$ , then by (2),  $x_k$  exists and  $x_k \in B_\nu(0) \subset U'_d \forall k_0$  and  $\forall k \geq k_0$ . The proof of Proposition 1, which is entirely analogous to the proof of the Liapunov theorem in the continuous-time case, implies that the open ball with radius  $\nu'$  centered at 0,  $B_{\nu'}(0)$  belongs to the region of attraction.

**3. A sufficient condition for uniform asymptotic stability.** In this section, a sufficient condition for uniform asymptotic stability of a continuous-time system is proposed. In classical Liapunov theory, the time derivative of the Liapunov function  $V$  along the solutions of the system is required to be negative definite. In the present

case, the derivative of the  $V$ -function may have positive and negative values. The theorem requires only the existence of a positive definite and decrescent  $V$ -function and a sequence of times  $t_k^*$  such that the values of this  $V$ -function decrease when evaluated along the solutions at  $t_k^*$ .

Consider

$$(3) \quad \dot{x}(t) = f(x, t)$$

with  $f : W_c \times \mathbb{R} \rightarrow \mathbb{R}^n$ ,  $W_c$  open,  $W_c \subset \mathbb{R}^n$ . Let  $f(0, t) = 0 \forall t \in \mathbb{R}$  and  $0 \in W_c$ . Furthermore we assume that conditions are imposed on (3) such that existence and uniqueness of its solutions are secured. These conditions are imposed on all the differential equations mentioned in the present paper, and of these conditions we single out the local Lipschitz condition. This local Lipschitz condition will be used in the course of the proof of the propositions and the theorems hereafter:  $f$  is locally Lipschitz, i.e., for  $\forall x \in W_c$ ,  $\exists$  a neighborhood  $\mathcal{N}(x) \subset W_c$ , such that the restriction  $f|_{\mathcal{N}(x) \times \mathbb{R}} \stackrel{\text{not}}{=} f|_{\mathcal{N}(x)}$  is Lipschitz with Lipschitz function  $l_x(t)$ . We assume that  $l_x(t)$  is bounded  $\forall t \in \mathbb{R}$ . We are now ready to state a lemma and Proposition 2.

LEMMA 1. *Let  $U \subset W_c$  be an open neighborhood of 0. Consider a closed ball  $\overline{B}_\mu(0) \subset U$ ; then  $\forall T > 0$ ,  $\exists \mu' > 0$  such that  $(x_0, t_0) \in B_{\mu'}(0) \times \mathbb{R}$  implies that  $x(t) \in B_\mu(0)$  for  $t \in [t_0, t_0 + T]$ . Here  $x(t)$  is the solution of (3) evaluated at  $t$  with initial condition  $x_0$  at  $t_0$ . (The solutions are assumed to exist over the considered time interval.)*

*Proof.* The proof of the lemma is omitted. It can be found in [3]. □

Remark 2. The proof of the lemma shows that  $\mu' = \mu e^{-KT}$  is an appropriate choice of  $\mu'$ .  $K$  is the Lipschitz constant of (3) on  $\overline{B}_\mu(0)$  [10, p. 70].

PROPOSITION 2. *Consider a function  $V : U \times \mathbb{R} \rightarrow \mathbb{R}$ , with  $U$  an open neighborhood of 0. We assume that the following additional conditions are satisfied.*

1. *Condition 1.  $V(x, t)$  is positive definite and decrescent; i.e.,  $V(0, t) = 0 \forall t$  and  $\forall x \in U : \alpha_V(\|x\|) \leq V(x, t) \leq \beta_V(\|x\|)$ . The functions  $\alpha_V(\cdot) : \mathbb{R}^+ \rightarrow \mathbb{R}^+$  and  $\beta_V(\cdot) : \mathbb{R}^+ \rightarrow \mathbb{R}^+$  are strictly increasing continuous functions passing through the origin.*

2. *Condition 2. There exists an increasing sequence of times  $t_k^*$  ( $k \in \mathbb{Z}$ ) with  $t_k^* \rightarrow \infty$  as  $k \rightarrow \infty$  and  $t_k^* \rightarrow -\infty$  as  $k \rightarrow -\infty$ ,  $\exists$  finite  $T > 0 : t_{k+1}^* - t_k^* \leq T (\forall k \in \mathbb{Z})$ ; there exists a function  $\gamma(\cdot) : \mathbb{R}^+ \rightarrow \mathbb{R}^+$  which is continuous, strictly increasing, and passing through the origin and an open set  $U' \subset U \cap W_c$  which contains the origin such that  $\forall k \in \mathbb{Z} \forall x(t_k^*) \in U' \setminus \{0\}$ :*

$$(4) \quad V(x(t_{k+1}^*), t_{k+1}^*) - V(x(t_k^*), t_k^*) \leq -\gamma(\|x(t_k^*)\|) < 0,$$

where  $x(t_{k+1}^*)$  is the solution of (3) at  $t_{k+1}^*$  with initial condition  $x(t_k^*)$  at  $t_k^*$ .

Then the equilibrium point  $x = 0$  of (3) is locally uniformly asymptotically stable.

*Proof.* By the continuous-time system  $\dot{x} = f(x, t)$  and the sequence  $t_k^*$ , we define the discrete-time system  $x_{k+1} = g(x_k, k)$  with  $g(x_k, k) = x(t_{k+1}^*) \forall x_k \in U' = W_d$ . Here  $x(t_{k+1}^*)$  is the solution of (3) at  $t_{k+1}^*$  with initial condition  $x(t_k^*) = x_k$  at  $t_k^*$ . The continuous times  $t_k^*$  are identified with the discrete times  $k (\forall k \in \mathbb{Z})$  and the state  $x(t_k^*)$  is equal to  $x_k (\forall k \in \mathbb{Z})$ .

Uniform asymptotic stability of  $x_{k+1} = g(x_k, k)$ . Condition 1 and Condition 2 of Proposition 2 imply that Condition 1 and Condition 2 of Proposition 1 are fulfilled with  $W_d = U_d = U'_d = U'$ , which implies local uniform asymptotic stability of  $x_{k+1} = g(x_k, k)$ .

We are set to prove local uniform asymptotic stability of (3). First uniform stability will be established and then uniform convergence will be established.

*Uniform stability of  $\dot{x} = f(x, t)$ .* Consider a closed ball  $\overline{B}_\epsilon(0)$  centered at 0 and radius  $\epsilon$  small enough such that  $\overline{B}_\epsilon(0) \subset U'$ . Let  $K$  be the Lipschitz constant of (3) on  $\overline{B}_\epsilon(0)$ . Define  $\epsilon' := \epsilon e^{-KT}$ . Define  $\delta' := \beta_V^{-1}(\alpha_V(\epsilon'))$  and  $\delta'' := e^{-KT}\delta'$ .

Consider the open ball  $B_{\delta''}(0)$ . For all  $(x(t_0), t_0) \in B_{\delta''}(0) \times \mathbb{R}$ , there exists a  $k_0 \in \mathbb{Z}$  such that  $t_{k_0}^* - t_0 < T$ . By Lemma 1 and Remark 2 with  $\mu = \delta'$  and  $\mu' = \delta''$ , one obtains that  $\forall t \in [t_0, t_{k_0}^*]$ ,  $t - t_0 < T$  which implies that

$$(5) \quad \|x(t)\| < \delta' = \delta'' e^{KT} < \epsilon \quad \forall t \in [t_0, t_{k_0}^*]$$

and  $\|x(t_{k_0}^*)\| < \delta'$ .

By Proposition 1 and Remark 1 with  $\nu = \epsilon'$  and  $\nu' = \delta'$ , it is clear that  $\forall n \in \mathbb{N}$ :  $x_{k_0+n}$  exists and  $\|x_{k_0+n}\| < \epsilon'$ , where  $x_{k_0+n}$  is the solution of  $x_{k+1} = g(x_k, k)$  at  $k_0 + n$  with initial condition  $x_{k_0} = x(t_{k_0}^*)$  at  $k_0$ .

For all  $n \in \mathbb{N}$ ,  $x_{k_0+n}$  equals  $x(t_{k_0+n}^*)$ , where  $x(t_{k_0+n}^*)$  is the solution of (3) at  $t_{k_0+n}^*$  with initial condition  $x(t_{k_0}^*) = x_{k_0}$  at  $t_{k_0}^*$ . This implies that  $\forall n \in \mathbb{N} : \|x(t_{k_0+n}^*)\| = \|x_{k_0+n}\| < \epsilon'$ .

Notice that  $\forall t \geq t_{k_0}^*, \exists n \in \mathbb{N}$  such that  $t - t_{k_0+n}^* < T$ . Since  $\|x(t_{k_0+n}^*)\| < \epsilon' = \epsilon e^{-KT} \forall n \in \mathbb{N}$ , Lemma 1 and Remark 2 with  $\mu = \epsilon$  and  $\mu' = \epsilon'$  imply that

$$(6) \quad \|x(t)\| < \epsilon \quad \forall t \geq t_{k_0}^*.$$

By (5) and (6),  $\|x(t)\| < \epsilon \forall t \geq t_0$  when  $(x(t_0), t_0) \in B_{\delta''}(0) \times \mathbb{R}$ .

Uniform stability of (3) is established when

$$\forall \epsilon_c > 0, \exists \delta(\epsilon_c) > 0 : \|x(t_0)\| < \delta(\epsilon_c) \Rightarrow \|x(t)\| < \epsilon_c \quad \forall t, t_0 \text{ such that } t \geq t_0.$$

If  $\overline{B}_{\epsilon_c} \subset U'$ , then take  $\epsilon = \epsilon_c$ . If  $\overline{B}_{\epsilon_c}(0) \not\subset U'$ , then take  $\epsilon$  small enough such that  $\overline{B}_\epsilon(0) \subset U'$ . Simply take  $\delta(\epsilon_c) = \delta'' = e^{-KT}\beta_V^{-1}(\alpha_V(\epsilon e^{-KT}))$ .

*Uniform convergence of  $\dot{x} = f(x, t)$ .* We prove the existence of an  $\epsilon_1 > 0$  such that  $\forall \epsilon_2 > 0$ , there exists a  $T(\epsilon_2) \geq 0$  such that  $\|x(t_0)\| < \epsilon_1$  ( $t_0$  arbitrary) implies  $\|x(t)\| < \epsilon_2 \forall t \geq t_0 + T(\epsilon_2)$ . Here  $x(t)$  is the solution of (3) with initial condition  $x(t_0)$  at  $t_0$ .

Take  $\epsilon$  small enough such that  $\overline{B}_\epsilon(0) \subset U'$ . Let  $K$  be the Lipschitz constant of (3) on  $\overline{B}_\epsilon(0)$ . Take  $\epsilon_1 = e^{-KT}\beta_V^{-1}(\alpha_V(\epsilon e^{-KT}))$ . For all  $(x(t_0), t_0) \in B_{\epsilon_1}(0) \times \mathbb{R}$ , there exists a  $k_0 \in \mathbb{Z}$  such that  $t_{k_0}^* - t_0 < T$ . Since  $\|x(t_0)\| < \epsilon_1$ , by Lemma 1 and Remark 2 with  $\mu = \epsilon_1 e^{KT}$  and  $\mu' = \epsilon_1$ , one obtains that

$$(7) \quad \|x(t_{k_0}^*)\| < \epsilon_1 e^{KT} = \beta_V^{-1}(\alpha_V(\epsilon e^{-KT})).$$

Take  $\epsilon_{1d} := \beta_V^{-1}(\alpha_V(\epsilon))$ . By the convergence property of  $x_{k+1} = g(x_k, k)$ , Proposition 1 and Remark 1 with  $\nu = \epsilon$  and  $\nu' = \epsilon_{1d}$  imply that  $\forall \epsilon_{2d} : \exists k(\epsilon_{2d})$  such that  $\|x_{k_0}\| < \epsilon_{1d}$  implies that  $\|x_k\| < \epsilon_{2d} \forall k \geq k_0 + k(\epsilon_{2d})$ .

Since by (7)  $\|x_{k_0}\| = \|x(t_{k_0}^*)\| < \epsilon_{1d}$ , one obtains by taking  $\epsilon_{2d} = \epsilon_2 e^{-KT}$  that  $\forall k \geq k_0 + k(\epsilon_{2d})$  that

$$(8) \quad \|x_k\| = \|x(t_k^*)\| < \epsilon_2 e^{-KT}.$$

By Lemma 1 and Remark 2 with  $\mu = \epsilon_2$  and  $\mu' = \epsilon_2 e^{-KT}$

$$(9) \quad \|x(t)\| < \epsilon_2 \quad \forall t \geq t_{k_0+k(\epsilon_{2d})}^*.$$

Since  $t_{k_0}^* - t_0 < T$  and since  $t_{k_0+k(\epsilon_{2d})}^* - t_{k_0}^* \leq k(\epsilon_{2d})T$ ,  $t_{k_0+k(\epsilon_{2d})}^* - t_0 < (k(\epsilon_{2d}) + 1)T$ . Therefore  $\|x(t)\| < \epsilon_2 \forall t \geq t_0 + T(\epsilon_2)$  with  $T(\epsilon_2) := (k(\epsilon_2 e^{-KT}) + 1)T$ . This implies uniform convergence to the origin and therefore also uniform asymptotic stability of (3).  $\square$

*Remark 3.* The proof of Proposition 2 shows that when  $\overline{B}_\epsilon(0) \subset U'$  that  $B_{\delta(\epsilon)}(0)$  with  $\delta(\epsilon) = e^{-KT} \beta_V^{-1}(\alpha_V(\epsilon e^{-KT}))$  belongs to the region of attraction of (3).

*Remark 4.* If Condition 2 of Proposition 2 is replaced by the condition that there exists an increasing sequence of times  $t_k^*$  ( $k \in \mathbb{Z}$ ) with  $t_k^* \rightarrow \infty$  as  $k \rightarrow \infty$  and  $t_k^* \rightarrow -\infty$  as  $k \rightarrow -\infty$ ,  $\exists$  finite  $T > 0 : t_{k+1}^* - t_k^* \leq T$  ( $\forall k \in \mathbb{Z}$ ), there exists an open set  $U' \subset U \cap W_c$  which contains the origin such that  $\forall k \in \mathbb{Z} \forall x(t_k^*) \in U' \setminus \{0\}$ ,

$$V(x(t_{k+1}^*), t_{k+1}^*) - V(x(t_k^*), t_k^*) \leq 0,$$

then the equilibrium point  $x = 0$  of (3) is locally uniformly stable.

**4. Homogeneous systems.** Proposition 2 introduces a sufficient condition for uniform asymptotic stability of a dynamical system. Because of Condition 2, it may be hard in general to verify uniform asymptotic stability by means of this proposition. This section and section 5 may be seen as an elaboration of the previous one. When considering homogeneous systems, we show that Condition 2 of Proposition 2 may be replaced by a condition *independent of the flow*.

**DEFINITION 1.** *The system  $\dot{x}(t) = f(x, t)$  with  $x = (x_1, \dots, x_n)^T$  is homogeneous of order  $\tau$  and with dilation  $\delta(s, x) = (s^{r_1}x_1, \dots, s^{r_n}x_n)^T$  ( $\forall i \in \{1, \dots, n\} : 0 < r_i < \infty$ ) if for each  $i \in \{1, \dots, n\}$*

$$(10) \quad \forall x \in \mathbb{R}^n, \forall t, \forall s \geq 0 : f_i(s^{r_1}x_1, \dots, s^{r_n}x_n, t) = s^{\tau+r_i} f_i(x_1, \dots, x_n, t).$$

**DEFINITION 2.** *The homogeneous  $p$ -norm  $\rho_p$  with dilation  $\delta(s, x)$  is a continuous map from  $\mathbb{R}^n$  to  $\mathbb{R}^+$ ,  $x \rightarrow \rho_p(x)$  such that*

$$(11) \quad \rho_p(x) := \left( \sum_{i=1}^n |x_i|^{\frac{p}{r_i}} \right)^{\frac{1}{p}}$$

with  $p \in \mathbb{R}_0^+$ .

*Remark 5.* Calling the function  $\rho_p$  a “norm” is a misnomer. In general  $\rho_p$  does not satisfy the triangle inequality or the scale property.

**PROPOSITION 3.** *Consider the homogeneous system  $\dot{x}(t) = f(x, t)$  of order  $\tau > 0$  and with dilation  $\delta(s, x) = (s^{r_1}x_1, \dots, s^{r_n}x_n)^T$ .  $f$  is locally Lipschitz, i.e.,  $\forall x, \exists$  neighborhood  $\mathcal{N}(x)$  such that the restriction  $f|_{\mathcal{N}(x)}$  is Lipschitz with Lipschitz function  $l_x(t)$  and  $l_x(t)$  is bounded  $\forall t \in \mathbb{R}$ .*

*If there exists an increasing sequence of times  $t_k^*$  ( $k \in \mathbb{Z}$ ) with  $t_k^* \rightarrow \infty$  as  $k \rightarrow \infty$  and  $t_k^* \rightarrow -\infty$  as  $k \rightarrow -\infty$ ,  $\exists$  finite  $T > 0 : t_{k+1}^* - t_k^* \leq T$  ( $\forall k \in \mathbb{Z}$ ) and  $\exists K_1 > 0$  such that  $\forall k$  and  $\forall x$  with  $\rho_r(x) = 1$  and  $r > \max\{r_1, \dots, r_n\}$*

$$(12) \quad \frac{\partial V}{\partial x}(x) \int_{t_k^*}^{t_{k+1}^*} f(x, t) dt \leq -K_1 T,$$

where

1.  $V(x)$  is a positive definite continuous homogeneous function, i.e.,

$$(13) \quad \forall x \in \mathbb{R}^n, \forall s \geq 0 : V(s^{r_1}x_1, \dots, s^{r_n}x_n) = s^l V(x_1, \dots, x_n)$$

for some  $l > 0$ , and

2.  $\frac{\partial V}{\partial x}(x) := (\frac{\partial V}{\partial x_1}(x), \dots, \frac{\partial V}{\partial x_n}(x))$  is locally Lipschitz on  $\mathbb{R}^n$ ;  
 then  $\dot{x} = f(x, t)$  is locally uniformly asymptotically stable.

*Proof.* Before starting the main part of the proof, we introduce some notation and we derive a number of inequalities to be used later on in the proof.

Define  $S_\beta := \{x \in \mathbb{R}^n : |x_i| \leq \beta, \forall i \in \{1, \dots, n\}\}$  ( $\beta \in \mathbb{R}_0^+$ ). If we use the norm  $\|x\|_{\max} = \max_{1 \leq i \leq n} |x_i|$ , then  $\forall x \in S_\beta : \|x\|_{\max} \leq \beta$ . The max-norm is denoted as  $\|\cdot\|_{\max}$  whereas the Euclidean norm is denoted as  $\|\cdot\|$ .

By the local Lipschitz property of  $\dot{x} = f(x, t)$ , it is Lipschitz on  $S_\beta$  with a Lipschitz constant which we denote as  $K_{f\beta}$ . Therefore,  $\forall x, y \in S_\beta$  and  $\forall i \in \{1, \dots, n\}$

$$(14) \quad |f_i(x, t) - f_i(y, t)| \leq \|f(x, t) - f(y, t)\|_{\max} \leq K_{f\beta} \|x - y\|_{\max}.$$

The set  $\{x \in \mathbb{R}^n | \rho_r(x) = 1\} \subset S_1$  and therefore if  $i \in \{1, \dots, n\}$  and  $\rho_r(x) = 1$ , then

$$(15) \quad |f_i(x, t)| \leq \|f(x, t)\|_{\max} \leq K_f \|x\|_{\max} \leq K_f$$

with  $K_f := K_{f1}$ . By the same argument, we obtain that  $\forall x, y \in S_\beta$

$$(16) \quad \left\| \frac{\partial V}{\partial x}(x) - \frac{\partial V}{\partial x}(y) \right\|_{\max} \leq K_{V\beta} \|x - y\|_{\max},$$

where  $K_{V\beta}$  is the Lipschitz constant of  $\frac{\partial V}{\partial x}$  on  $S_\beta$ . When  $\rho_r(x) = 1$ , then

$$(17) \quad \left\| \frac{\partial V}{\partial x}(x) \right\|_{\max} \leq K_V$$

with  $K_V := K_{V1}$ . The estimates (14), (15), (16), and (17) will be used in the following.

**I. Evolution of the V-function with respect to the sequence  $t_k^*$ .** The time-derivative of  $V(x)$  along the solutions of the system  $\dot{x}(t) = f(x, t)$  is given by

$$(18) \quad \dot{V}(x, t) = \frac{\partial V}{\partial x}(x) \dot{x} = \frac{\partial V}{\partial x}(x) f(x, t).$$

Consider

$$(19) \quad \Delta V(t_{k+1}^*, t_k^*) := \int_{t_k^*}^{t_{k+1}^*} \dot{V}(x, t) dt = \int_{t_k^*}^{t_{k+1}^*} \frac{\partial V}{\partial x}(x(t)) f(x(t), t) dt,$$

which may be rewritten as

$$(20) \quad \int_{t_k^*}^{t_{k+1}^*} \frac{\partial V}{\partial x}(x(t_k^*)) f(x(t_k^*), t) dt$$

$$(21) \quad + \int_{t_k^*}^{t_{k+1}^*} \left( \frac{\partial V}{\partial x}(x(t)) f(x(t), t) - \frac{\partial V}{\partial x}(x(t_k^*)) f(x(t_k^*), t) \right) dt.$$

In order to evaluate (19), we successively estimate (20) and (21).

**II. Estimate of (20).** In order to invoke the homogeneity properties of  $\dot{x} = f(x, t)$ , we spell out (20) as

$$(22) \quad \sum_{i=1}^n \int_{t_k^*}^{t_{k+1}^*} \frac{\partial V}{\partial x_i}(x(t_k^*)) f_i(x(t_k^*), t) dt.$$

By partially differentiating each member of (13) with respect to  $x_i$ , one obtains that  $\forall x \in \mathbb{R}^n \setminus \{0\}$ ,  $\forall i \in \{1, \dots, n\}$ , and  $\forall s \geq 0$

$$(23) \quad \frac{\partial V}{\partial x_i}(s^{r_1} x_1, \dots, s^{r_n} x_n) = s^{l-r_i} \frac{\partial V}{\partial x_i}(x_1, \dots, x_n).$$

By (23) and the homogeneity of  $f(x, t)$ , one obtains that (22) can be written as

$$(24) \quad \rho_r^{\tau+l}(x(t_k^*)) \int_{t_k^*}^{t_{k+1}^*} \sum_{i=1}^n \frac{\partial V}{\partial x_i}(\delta(\rho_r^{-1}(x(t_k^*)), x(t_k^*))) f_i(\delta(\rho_r^{-1}(x(t_k^*)), x(t_k^*)), t) dt$$

or as

$$(25) \quad \rho_r^{\tau+l}(x(t_k^*)) \frac{\partial V}{\partial x}(\delta(\rho_r^{-1}(x(t_k^*)), x(t_k^*))) \int_{t_k^*}^{t_{k+1}^*} f(\delta(\rho_r^{-1}(x(t_k^*)), x(t_k^*)), t) dt.$$

Since  $\delta(\rho_r^{-1}(x(t_k^*)), x(t_k^*))$  has a homogeneous norm equal to 1, (25) implies by (12) that

$$(26) \quad \int_{t_k^*}^{t_{k+1}^*} \frac{\partial V}{\partial x}(x(t_k^*)) f(x(t_k^*), t) dt \leq -K_1 T \rho_r^{\tau+l}(x(t_k^*)).$$

The inequality (26) will force  $\Delta V(t_{k+1}^*, t_k^*)$  to be negative definite. We show that (21) cannot account for a sign change when the initial state  $x(t_k^*)$  is taken close enough to the origin. To prove this, we estimate an upper bound for the absolute value of (21).

**III. Estimate of (21).** By (23) and the homogeneity of  $f(x, t)$ , (21) can be written as

$$(27) \quad \rho_r^{\tau+l}(x(t_k^*)) \int_{t_k^*}^{t_{k+1}^*} L_f V(\delta(\rho_r^{-1}(x(t_k^*)), x(t)), t) - L_f V(\delta(\rho_r^{-1}(x(t_k^*)), x(t_k^*)), t) dt,$$

where  $\frac{\partial V}{\partial x}(x) f(x, t) =: L_f V(x, t)$ .

In order to evaluate an upper bound for the absolute value of (27), we need the max-norm of

$$(28) \quad \delta(\rho_r^{-1}(x(t_k^*)), x(t)) - \delta(\rho_r^{-1}(x(t_k^*)), x(t_k^*)) = \delta(\rho_r^{-1}(x(t_k^*)), x(t) - x(t_k^*))$$

since we use the Lipschitz property of  $L_f V$ . In III.1 we estimate the norm of (28) and in III.2 we estimate the Lipschitz constant corresponding to  $L_f V$ . In III.3 we use these results to estimate the absolute value of (21).

**III.1. Estimate of the norm of (28).** Since

$$(29) \quad x_i(t) - x_i(t_k^*) = \int_{t_k^*}^t f_i(x(s), s) ds,$$

one obtains by the homogeneity of  $f(x, s)$  and by (15) that

$$(30) \quad |x_i(t) - x_i(t_k^*)| \leq \int_{t_k^*}^t \rho_r^{\tau+r_i}(x(s)) K_f ds.$$

Since  $\frac{\partial \rho_r}{\partial x_i}(s^{r_1} x_1, \dots, s^{r_n} x_n) = s^{1-r_i} \frac{\partial \rho_r}{\partial x_i}(x_1, \dots, x_n)$  and

$$(31) \quad \begin{aligned} \frac{d}{d\sigma} \rho_r(x(\sigma)) &= \sum_{i=1}^n \frac{\partial \rho_r}{\partial x_i}(x(\sigma)) f_i(x(\sigma), \sigma) \\ &= \rho_r^{\tau+1}(x(\sigma)) \sum_{i=1}^n \frac{\partial \rho_r}{\partial x_i}(y(\sigma)) f_i(y(\sigma), \sigma) \end{aligned}$$

with  $y(\sigma) = \delta(\rho_r^{-1}(x(\sigma)), x(\sigma))$ . By defining  $g(\sigma) := \sum_{i=1}^n \frac{\partial \rho_r}{\partial x_i}(y(\sigma)) f_i(y(\sigma), \sigma)$ ,

$$(32) \quad \frac{d}{d\sigma} \rho_r(x(\sigma)) = \rho_r^{\tau+1}(x(\sigma)) g(\sigma).$$

By direct substitution [6], it is clear that the solution of (32) equals

$$(33) \quad \rho_r(x(s)) = \frac{\rho_r(x(t_k^*))}{(1 - \tau \rho_r^\tau(x(t_k^*)) \int_{t_k^*}^s g(\sigma) d\sigma)^{\frac{1}{\tau}}}$$

under the assumption that

$$(34) \quad 1 - \tau \rho_r^\tau(x(t_k^*)) \int_{t_k^*}^s g(\sigma) d\sigma > 0.$$

Since  $r_i < r$ ,  $\frac{\partial \rho_r}{\partial x_i}$  is continuous on the set  $\{x \in \mathbb{R}^n \mid \rho_r(x) = 1\}$ . Since  $y(\sigma)$  belongs to this compact set ( $\rho_r(y(\sigma)) = 1$ ), the continuity of  $\frac{\partial \rho_r}{\partial x_i}$  and (15) imply boundedness of  $g(\sigma)$ . There exists a  $g_m > 0$  such that  $\forall \sigma: g(\sigma) \leq g_m$ . This implies that

$$(35) \quad \rho_r(x(s)) \leq \frac{\rho_r(x(t_k^*))}{(1 - \tau \rho_r^\tau(x(t_k^*)) (s - t_k^*) g_m)^{\frac{1}{\tau}}} < 2^{\frac{1}{\tau}} \rho_r(x(t_k^*))$$

when  $t \in [t_k^*, t_{k+1}^*]$  with  $t_{k+1}^* - t_k^* \leq T \forall k \in \mathbb{Z}$  and  $\rho_r(x(t_k^*)) < (2\tau g_m T)^{-\frac{1}{\tau}} =: \rho'$ . By (35), it is obvious that (30) implies that

$$(36) \quad |x_i(t) - x_i(t_k^*)| \leq 2^{\frac{\tau+r_i}{\tau}} K_f T \rho_r^{\tau+r_i}(x(t_k^*)).$$

Recall that

$$(37) \quad \|\delta(\rho_r^{-1}(x(t_k^*)), x(t)) - \delta(\rho_r^{-1}(x(t_k^*)), x(t_k^*))\|_{\max} = \max_{1 \leq i \leq n} \frac{|x_i(t) - x_i(t_k^*)|}{\rho_r^{r_i}(x(t_k^*))}.$$

Therefore, (36) and (37) imply the existence of a  $\tilde{K} > 0$  such that

$$(38) \quad \|\delta(\rho_r^{-1}(x(t_k^*)), x(t)) - \delta(\rho_r^{-1}(x(t_k^*)), x(t_k^*))\|_{\max} \leq \tilde{K} T \rho_r^\tau(x(t_k^*))$$

when  $\rho_r(x(t_k^*)) < \rho'$  and  $t \in [t_k^*, t_{k+1}^*]$ .

Having estimated the norm of (28), we estimate the Lipschitz constant of  $L_f V$  which will be used in the calculation of an upper bound for (21).

**III.2. Estimate of the Lipschitz constant.** Notice that

$$(39) \quad \begin{aligned} \delta(\rho_r^{-1}(x(t_k^*)), x(t)) &= \delta(\rho_r^{-1}(x(t_k^*)), x(t_k^*)) \\ &\quad + \delta(\rho_r^{-1}(x(t_k^*)), x(t)) - \delta(\rho_r^{-1}(x(t_k^*)), x(t_k^*)) \end{aligned}$$

and

$$(40) \quad \begin{aligned} \|\delta(\rho_r^{-1}(x(t_k^*)), x(t))\|_{\max} &\leq \|\delta(\rho_r^{-1}(x(t_k^*)), x(t_k^*))\|_{\max} \\ &\quad + \|\delta(\rho_r^{-1}(x(t_k^*)), x(t)) - \delta(\rho_r^{-1}(x(t_k^*)), x(t_k^*))\|_{\max} \end{aligned}$$

such that by (38)

$$(41) \quad \|\delta(\rho_r^{-1}(x(t_k^*)), x(t))\|_{\max} \leq 1 + \tilde{K} T \rho_r^\tau(x(t_k^*)) \leq 1 + \frac{\tilde{K}}{2\tau g_m} =: \tilde{\beta}$$

when  $t \in [t_k^*, t_{k+1}^*]$  and  $\rho_r(x(t_k^*)) < \rho' = (2\tau g_m T)^{-\frac{1}{\tau}}$ . Therefore, (41) implies that  $\delta(\rho_r^{-1}(x(t_k^*)), x(t)) \in S_{\tilde{\beta}}$ .

Since  $\forall x, y \in S_{\tilde{\beta}}$  and  $\forall t$

$$(42) \quad \begin{aligned} &L_f V(x, t) - L_f V(y, t) \\ &= \frac{\partial V}{\partial x}(x)(f(x, t) - f(y, t)) + \left( \frac{\partial V}{\partial x}(x) - \frac{\partial V}{\partial x}(y) \right) f(y, t), \end{aligned}$$

the boundedness of  $\frac{\partial V}{\partial x}$  and  $f$  on  $S_{\tilde{\beta}}$ , implied by (14) and (16), and the Lipschitz properties of  $\frac{\partial V}{\partial x}$  and  $f$  imply the existence of a Lipschitz constant  $K_{fV\tilde{\beta}}$  for  $L_f V$  on  $S_{\tilde{\beta}}$ . Therefore,  $\forall x, y \in S_{\tilde{\beta}}$ , and  $\forall t$

$$(43) \quad |L_f V(x, t) - L_f V(y, t)| \leq K_{fV\tilde{\beta}} \|x - y\|_{\max}.$$

**III.3. Estimate of (21).** By (38) and (43), one obtains that

$$(44) \quad \begin{aligned} &|L_f V(\delta(\rho_r^{-1}(x(t_k^*)), x(t)), t) - L_f V(\delta(\rho_r^{-1}(x(t_k^*)), x(t_k^*)), t)| \\ &\leq K_{fV\tilde{\beta}} \tilde{K} T \rho_r^\tau(x(t_k^*)). \end{aligned}$$

By (44), the absolute value of (27)–(21) is less than or equal to

$$(45) \quad K_{fV\tilde{\beta}} \tilde{K} T^2 \rho_r^{2\tau+l}(x(t_k^*))$$

when  $\rho_r(x(t_k^*)) < \rho'$ .

**IV. Estimate of (19).** By (26) and (45), (19) is less than or equal to

$$(46) \quad \rho_r^{\tau+l}(x(t_k^*)) T (-K_1 + \tilde{K} K_{fV\tilde{\beta}} T \rho_r^\tau(x(t_k^*))).$$

Define  $\rho := \min\{\rho', (\frac{K_1}{2K_{fV\tilde{\beta}}\tilde{K}T})^{\frac{1}{\tau}}\}$ . This implies by (46) that  $\forall x(t_k^*) \neq 0$  with  $\rho_r(x(t_k^*)) < \rho$ :

$$(47) \quad \Delta V(t_{k+1}^*, t_k^*) = V(x(t_{k+1}^*)) - V(x(t_k^*)) \leq -\frac{K_1 T}{2} \rho_r^{\tau+l}(x(t_k^*)),$$

where  $x(t_{k+1}^*)$  is the solution of  $\dot{x}(t) = f(x, t)$  at  $t_{k+1}^*$  with initial condition  $x(t_k^*)$  at  $t_k^*$ .

**V. Uniform asymptotic stability.** By (13), it is obvious that Condition 1 of Proposition 2 is fulfilled with  $U = \mathbb{R}^n$ . By (47), it is clear that Condition 2 of Proposition 2 is fulfilled with  $U' = \{x | \rho_r(x) < \rho\}$ . Therefore, Proposition 2 may be applied, which implies local uniform asymptotic stability of the homogeneous system  $\dot{x} = f(x, t)$ .  $\square$

*Remark 6.* Notice that  $r$  is taken to be larger than  $\max\{r_1, \dots, r_n\}$  (and not equal to  $\max\{r_1, \dots, r_n\}$ ) in order to avoid technical difficulties when taking the derivative of  $\sum_{i=1}^n |y_i|^{\frac{r}{r_i}}$  with respect to time.

**5. Uniform asymptotic stability of time-varying homogeneous systems.**

Having Proposition 3 available, it is now possible to establish that asymptotic stability of the averaged system of a time-varying homogeneous system implies local uniform asymptotic stability of the original time-varying homogeneous system. Because of the homogeneity and the order condition  $\tau > 0$ , this result is valid even when the system is not fast time-varying.

**THEOREM 1.** Consider the homogeneous system  $\dot{x}(t) = f(x, t)$  of order  $\tau > 0$  and with dilation  $\delta(s, x) = (s^{r_1}x_1, \dots, s^{r_n}x_n)^T$ .  $f$  is locally Lipschitz, i.e.,  $\forall x, \exists$  neighborhood  $\mathcal{N}(x)$  such that the restriction  $f|_{\mathcal{N}(x)}$  is Lipschitz with Lipschitz function  $l_x(t)$  and  $l_x(t)$  is bounded  $\forall t \in \mathbb{R}$ . If the following conditions hold:

Condition 1. The averaged system  $\dot{x}(t) = \bar{f}(x)$  is asymptotically stable, where

$$(48) \quad \bar{f}(x) := \lim_{T \rightarrow \infty} \frac{1}{2T} \int_{-T}^{+T} f(x, t) dt$$

is continuous in  $x$ ;

Condition 2. There exists a continuous function  $M : [0, +\infty[ \rightarrow [0, +\infty[$  with  $\lim_{\sigma \rightarrow \infty} \sigma^{-1}M(\sigma) = 0$  such that  $\forall t_1, t_2 \in \mathbb{R}$  ( $t_2 > t_1$ )

$$(49) \quad \left\| \int_{t_1}^{t_2} (f(x, t) - \bar{f}(x)) dt \right\| \leq M(t_2 - t_1)$$

when  $\rho_r(x) = 1$  with  $r > \max\{r_1, \dots, r_n\}$ .

Then  $\dot{x}(t) = f(x, t)$  is locally uniformly asymptotically stable.

*Proof.* The proof of the theorem is based on Proposition 3. Definition (48) of the averaged system  $\dot{x}(t) = \bar{f}(x)$  implies its homogeneity of order  $\tau$  with dilation  $\delta(s, x) = (s^{r_1}x_1, \dots, s^{r_n}x_n)^T$ . Definition (48) also implies that  $\bar{f}(0) = 0$ . By Condition 1, the homogeneous system  $\dot{x} = \bar{f}(x)$  is asymptotically stable. Let  $p$  be a positive integer and let  $l$  be a real number larger than  $p(\max_{1 \leq i \leq n} r_i)$ . Following Rosier [13], there exists a Liapunov function  $V : \mathbb{R}^n \rightarrow \mathbb{R}$  such that

- P1.  $V(x)$  is of class  $C^\infty$  in  $\mathbb{R}^n \setminus \{0\}$  and of class  $C^p$  in  $\mathbb{R}^n$ ;
- P2.  $V(0) = 0, V(x) > 0 \forall x \neq 0$  and  $V(x) \rightarrow +\infty$  as  $\|x\| \rightarrow +\infty$ ;
- P3.  $V$  is homogeneous:  $\forall x \in \mathbb{R}^n \setminus \{0\} : \forall s > 0 : V(s^{r_1}x_1, \dots, s^{r_n}x_n) = s^l V(x_1, \dots, x_n)$ ;
- P4.  $\forall x \neq 0 : \frac{\partial V}{\partial x}(x) \bar{f}(x) < 0$ .

By P1,  $\frac{\partial V}{\partial x}$  is continuous in  $\mathbb{R}^n$  and the continuity of  $\bar{f}(x)$  implies continuity of  $\frac{\partial V}{\partial x}(x) \bar{f}(x)$  on the compact set  $\{x | \rho_r(x) = 1\}$ . This implies by P4 that  $\exists \beta > 0$  such that  $\forall x$  with  $\rho_r(x) = 1$ ,

$$(50) \quad \frac{\partial V}{\partial x}(x) \bar{f}(x) \leq -\beta.$$

Take a  $t_0^\circ \in \mathbb{R}$  and  $\forall T'$  and  $\forall x$  with  $\rho_r(x) = 1$ :

$$(51) \quad \frac{\partial V}{\partial x}(x) \int_{t_0^\circ}^{t_0^\circ + T'} f(x, t) dt = T' \frac{\partial V}{\partial x}(x) \bar{f}(x) + \frac{\partial V}{\partial x}(x) \int_{t_0^\circ}^{t_0^\circ + T'} (f(x, t) - \bar{f}(x)) dt.$$

By the continuity of  $\frac{\partial V}{\partial x}$  on the compact set  $\{x | \rho_r(x) = 1\}$ ,  $\exists M_V > 0$  such that

$$(52) \quad \left\| \frac{\partial V}{\partial x}(x) \right\| \leq M_V$$

when  $\rho_r(x) = 1$ . One obtains by Condition 2 that

$$(53) \quad \frac{\partial V}{\partial x}(x) \int_{t_0^\circ}^{t_0^\circ + T'} f(x, t) dt \leq T' \frac{\partial V}{\partial x}(x) \bar{f}(x) + M_V M(T')$$

$$(54) \quad \leq -\beta T' + M_V M(T')$$

when  $\rho_r(x) = 1$ . By Condition 2,  $\lim_{T' \rightarrow \infty} \frac{M(T')}{T'} = 0$ , which implies that  $\exists T''$  such that  $\forall T' \geq T'' : \frac{M(T')}{T'} < \frac{\beta}{M_V}$ . Take such a  $T'$  and define a sequence of times  $t_k^* := t_0^* + kT'$ , then  $\forall k \in \mathbb{Z}$  and  $\forall x$  with  $\rho_r(x) = 1$ ,

$$(55) \quad \frac{\partial V}{\partial x}(x) \int_{t_k^*}^{t_{k+1}^*} f(x, t) dt \leq -K_1 T'$$

with  $K_1 = \beta - M_V \frac{M(T')}{T'} > 0$ .

With  $T = T'$  and  $t_k^* = t_0^* + kT'$ , (55) implies that (12) is satisfied and P3 implies that (13) is satisfied. With  $p \geq 2$ ,  $\frac{\partial V}{\partial x}$  is continuously differentiable and therefore also locally Lipschitz. By Proposition 3, one obtains local uniform asymptotic stability of the homogeneous system  $\dot{x} = f(x, t)$ .  $\square$

*Remark 7.* Theorem 1 is a result on asymptotic stability of a homogeneous system under a classical averaging condition, *without* requirements on the time-scale—typical for averaging results.

*Remark 8.* The proof of Theorem 1 is based on Proposition 3 and the crucial part of the proof of Proposition 3 is the negative definiteness of (19). This negative definiteness of (19) is guaranteed by the negative definiteness of (20) when (21) is sufficiently small. The expression (21) is small when, roughly, the variation of the flow  $x(t) - x(t_k^*)$  is small in comparison with  $x(t_k^*)$ . This proportional variation can be reduced by reducing the size of the right-hand side  $\rho_r^r(x(t_k^*))\tilde{K}T$  of (38). A first way of achieving this reduction is by a time-scale transformation. It is clear that when the homogeneous system  $\dot{x} = f(x, t)$  satisfies the conditions of Theorem 1 with  $T = T_f$ , then  $\forall \alpha > 0: \dot{x} = f(x, \alpha t)$  satisfies the conditions of Theorem 1 with  $T = \frac{T_f}{\alpha}$ . By increasing  $\alpha$ , i.e., by decreasing  $T = \frac{T_f}{\alpha}$ ,  $\rho_r^r(x(t_k^*))\tilde{K}T$  can be made arbitrarily small. For any fixed time-scale, there is still a second possibility of reducing the size of  $\rho_r^r(x(t_k^*))\tilde{K}T$  by reducing  $\rho_r(x(t_k^*))$ , i.e., by starting at initial conditions close enough to the origin. This is the technique that leads to Theorem 1.

In [4], where exponential stability is considered, and in [11], where homogeneous systems of order  $\tau = 0$  are considered, there is a different situation. The effect of a decreasing proportional variation of the flow when starting close enough to the origin is not available. The only way to obtain a small proportional variation is by making  $\alpha$  sufficiently large. This explains why the averaging results in [4] and [11] imply uniform asymptotic stability of *fast* time-varying systems.

**6. A counterexample and an example.** The averaging result of Theorem 1, for not necessarily fast time-varying systems, is valid for homogeneous systems of order  $\tau > 0$ . As explained in Remark 8, the averaging result is not valid for homogeneous systems with an order  $\tau = 0$ .

This is easily illustrated. Consider the linear time-varying system  $\dot{x}(t) = A(t)x(t)$  [10, p. 144] with

$$(56) \quad A(t) = \begin{pmatrix} -1 + 1.5 \cos^2 t & 1 - 1.5 \sin t \cos t \\ -1 - 1.5 \sin t \cos t & -1 + 1.5 \sin^2 t \end{pmatrix}.$$

The averaged system  $\dot{x}(t) = \bar{A}x(t)$  with

$$(57) \quad \bar{A} = \begin{pmatrix} -0.25 & 1 \\ -1 & -0.25 \end{pmatrix}$$

is asymptotically stable but the original time-varying system  $\dot{x}(t) = A(t)x(t)$  is unstable with transition matrix

$$(58) \quad \Phi(t, 0) = \begin{pmatrix} e^{0.5t} \text{cost} & e^{-t} \text{sint} \\ -e^{0.5t} \text{sint} & e^{-t} \text{cost} \end{pmatrix}.$$

Since  $\bar{A}$  is Hurwitz, there exists a positive definite matrix  $P$  such that  $\bar{A}^T P + P \bar{A}$  is negative definite. Consider the homogeneous system  $\dot{x}(t) = \|x(t)\|A(t)x(t)$  with positive order. The averaged system  $\dot{x}(t) = \|x(t)\|\bar{A}x(t)$  is asymptotically stable since along its flow the derivative of  $x^T P x$  equals  $\|x\|x^T(\bar{A}^T P + P \bar{A})x$ , which is negative definite. By Theorem 1, asymptotic stability of  $\dot{x}(t) = \|x(t)\|\bar{A}x(t)$  implies local uniform asymptotic stability of  $\dot{x}(t) = \|x(t)\|A(t)x(t)$ .

**7. Semiglobal uniform asymptotic stability.** The conditions of Theorem 1 imply local uniform asymptotic stability of the homogeneous time-varying system  $\dot{x} = f(x, t)$  with order  $\tau > 0$ . These conditions also imply that  $\forall \alpha > 0$ :  $\dot{x} = f(x, \alpha t)$  is locally uniformly asymptotically stable. No global stability property is obtained. In the present section, we prove that the region of attraction of the homogeneous system  $\dot{x} = f(x, \alpha t)$  depends on  $\alpha$ . We show that the bounded region of attraction increases when  $\alpha$  increases and grows unbounded as  $\alpha$  goes to infinity.

**THEOREM 2.** *The homogeneous system  $\dot{x} = f(x, \alpha t)$  of order  $\tau > 0$ , where  $\dot{x} = f(x, t)$  satisfies all the conditions of Theorem 1, is semiglobally uniformly asymptotically stable, i.e.,  $\forall R > 0, \exists \alpha_R > 0$ , and also a class  $\mathcal{KL}$ -function  $\beta_R(\cdot, \cdot)$  such that  $\forall x_0$  with  $\rho_r(x_0) < R, \forall t_0, \forall t \geq t_0$*

$$(59) \quad \rho_r(x_{\alpha_R}(t, t_0, x_0)) \leq \beta_R(\rho_r(x_0), t - t_0).$$

Here  $x_{\alpha_R}(t, t_0, x_0)$  is the solution at  $t$  of the homogeneous system  $\dot{x} = f(x, \alpha_R t)$  with initial condition  $x_0$  at  $t_0$ .

*Proof.* The conditions imposed by Theorem 1 on  $\dot{x} = f(x, t)$  are satisfied, which implies local uniform asymptotic stability of  $\dot{x} = f(x, t)$ . This is equivalent to the existence of a  $\rho > 0$  and a class  $\mathcal{KL}$ -function  $\beta_\rho(\cdot, \cdot)$  such that  $\forall t_0, \forall t \geq t_0, \forall x_0$  with  $\rho_r(x_0) < \rho$

$$(60) \quad \rho_r(x_1(t, t_0, x_0)) \leq \beta_\rho(\rho_r(x_0), t - t_0),$$

where  $x_1(t, t_0, x_0)$  denotes the solution at  $t$  of  $\dot{x} = f(x, t)$  with initial condition  $x_0$  at  $t_0$ . We denote the solution of  $\dot{x} = f(x, \alpha t)$  with initial condition  $x_0$  at  $t_0$  as  $x_\alpha(t, t_0, x_0)$ . The solutions of  $\dot{x} = f(x, t)$  and  $\dot{x} = f(x, \alpha t)$  are related, i.e.,  $\forall \alpha > 0, \forall t_0$ , and  $\forall t \geq t_0$

$$(61) \quad x_\alpha(t, t_0, x_0) = \delta(\sqrt[\tau]{\alpha}, x_1(\alpha t, \alpha t_0, \delta^{-1}(\sqrt[\tau]{\alpha}, x_0))).$$

Define  $\alpha_R := (\frac{R}{\rho})^\tau$ . For all  $x_0$  with  $\rho_r(x_0) < R, \forall t_0, \forall t \geq t_0$ , by (61)

$$(62) \quad \rho_r(x_{\alpha_R}(t, t_0, x_0)) = \sqrt[\tau]{\alpha_R} \rho_r(x_1(\alpha_R t, \alpha_R t_0, \delta^{-1}(\sqrt[\tau]{\alpha_R}, x_0))).$$

Since  $\rho_r(\delta^{-1}(\sqrt[\tau]{\alpha_R}, x_0)) < \rho$ , applying (60) to the right-hand side of (62) implies that for all  $x_0$  with  $\rho_r(x_0) < R, \forall t_0, \forall t \geq t_0$

$$(63) \quad \rho_r(x_{\alpha_R}(t, t_0, x_0)) \leq \sqrt[\tau]{\alpha_R} \beta_\rho(\rho_r(\delta^{-1}(\sqrt[\tau]{\alpha_R}, x_0)), \alpha_R(t - t_0)) = \beta_R(\rho_r(x_0), t - t_0)$$

with the obvious definition of  $\beta_R(\cdot, \cdot)$ . □

**8. Conclusions.** Proposition 2 gives a sufficient condition for uniform asymptotic stability of a differential equation. This result is related to the result of Narendra and Annaswamy [12], but negative semidefiniteness on  $\dot{V}(x, t)$  is dispensed with.

This result is useful for the investigation of uniform asymptotic stability of homogeneous systems with order  $\tau > 0$ . More precisely, averaging becomes a useful tool for studying uniform asymptotic stability: asymptotic stability of the averaged system implies local uniform asymptotic stability of the original time-varying system. It is important that this result is *not restricted to fast time-varying systems*. The region of attraction of  $\dot{x} = f(x, \alpha t)$  increases with increasing  $\alpha$ . The uniform asymptotic stability is semiglobal since by taking  $\alpha$  large enough, every bounded region of attraction can be guaranteed.

Comparing these results with the results of M'Closkey and Murray [11], the following should be noted:

1. We are dealing with homogeneous systems of order  $\tau > 0$  while M'Closkey and Murray are dealing with homogeneous systems of order  $\tau = 0$ .

2. Because of the order  $\tau > 0$ , asymptotic stability of the averaged system implies asymptotic stability, not exponential stability, of the original time-varying system. M'Closkey and Murray consider homogeneous systems of order  $\tau = 0$  and therefore they are able to conclude exponential stability, with respect to the homogeneous norm, of the original time-varying system.

3. We obtain local asymptotic stability results for the homogeneous system  $\dot{x} = f(x, \alpha t)$  with order  $\tau > 0$  for every  $\alpha > 0$ . By setting  $\sigma = \alpha t$  and  $\epsilon = \frac{1}{\alpha}$ ,  $\dot{x} = f(x, \alpha t)$  is equivalent to  $\dot{x} = \epsilon f(x, \sigma)$ . M'Closkey and Murray deal with homogeneous systems  $\dot{x} = \epsilon f(x, \sigma, \epsilon)$  of order  $\tau = 0$  and therefore, the stability results are valid only for  $\epsilon$  sufficiently small.

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