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Exponential Stability of Nonlinear Time-Varying Differential Equations and Partial Averaging*

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Abstract. In this paper we formulate, within the Liapunov framework, a sufficient condition for exponential stability of a differential equation. This condition gives rise to a new averaging result referred to as "partial averaging": exponential stability of a system $\dot{x}(t) = f(x, t, \alpha t)$, with α sufficiently large, is implied by exponential stability of a time-varying system $\dot{x}(t) = \bar{f}(x, t)$.

Key words. Differential equations, Exponential stability, Liapunov stability, Averaging, Circle criterion.

1. Introduction

The stability analysis of time-varying systems $\dot{x}(t) = f(x(t), t)$ being in general harder than the stability analysis of time-invariant systems, several approaches have been proposed in the literature to reduce its stability analysis to the analysis of related time-invariant systems [1], [2], [7]–[10]. Averaging is the most popular of these techniques. Exponential stability of the (time-invariant) averaged system $\dot{x}(t) = \bar{f}(x(t))$ implies exponential stability of the original (time-varying) system if the time-variation of the original system is sufficiently fast [1], [2], [9].

The averaging result proposed in [2] and [9] can be generalized in two ways. In [10] it has been observed that the fast time-variation hypothesis necessary for averaging results can be replaced by a homogeneity assumption on the vectorfield.

In the present paper we prove a second generalization of the averaging result in [2]: exponential stability of a system $\dot{x}(t) = f(x(t), t, \alpha t)$, with α sufficiently large, is implied by exponential stability of the *partially* averaged system $\dot{x}(t) = \bar{f}(x(t), t)$.

The partial averaging result of this paper (Section 7) is an extension of the averaging result of Hale [7, pp. 190–195]. Hale [7] discusses differential equations $\dot{x} = \varepsilon f(x, t) + \varepsilon h(\varepsilon t, x)$, with ε sufficiently small. By setting $\alpha = 1/\varepsilon$, the averaging result

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in [7] provides a tool for studying systems arising as $\dot{x} = f(x, \alpha t) + h(x, t)$, with α sufficiently large. These results are restricted to the case with $h(x, t)$ time-periodic and $f(x, t)$ almost time-periodic. In the present paper more general systems are studied arising as $\dot{x} = f(x, t, \alpha t)$ with no periodicity or almost periodicity conditions involved.

The results of this paper are not simply extensions of the results established in [7] and [9]: they also rely on a different proof technique. The technique does not involve a state transformation [7], [9], but rather a generalization of the technique first developed in [2]. First, exponential stability of the partially averaged system $\dot{x}(t) = \bar{f}(x, t)$ implies, by the converse theorem of Liapunov for exponential stability, the existence of a Liapunov function $V(x, t)$. The derivative of this Liapunov function along the flow of the partially averaged system is negative definite. Second, this $V(x, t)$ is used to establish exponential stability of $\dot{x} = f(x, t, \alpha t)$ for α sufficiently large. In general, the derivative of $V(x, t)$ along the flow of $\dot{x} = f(x, t, \alpha t)$ takes positive as well as negative values which precludes the use of a classical Liapunov theorem [9], [11]. For α sufficiently large, we prove that $V(x, t)$ decreases stepwise along the trajectories of $\dot{x}(t) = f(x(t), t, \alpha t)$. This stepwise decrease of $V(x, t)$ implies exponential stability of $\dot{x}(t) = f(x(t), t, \alpha t)$.

In the organization of this paper, the theoretical results on partial averaging are formulated by means of three theorems (i.e., Theorems 2–4). Theorem 4 proves that exponential stability of the partially averaged system implies exponential stability of the original system $\dot{x}(t) = f(x(t), t, \alpha t)$. The use of this result is illustrated by means of a linear example arising in system identification, and by a specific nonlinear example.

We also illustrate the use of the partial averaging technique by means of a generalization of the circle criterion. This generalization is based on Theorem 2 (in order to avoid additional smoothness conditions it is not based on Theorem 4) which illustrates that Theorem 2 is not only useful in proving Theorems 3 and 4, but is useful in its own.

2. Exponential Stability

In this section we recall the main theorem for exponential stability of [2] since it will be a tool in proving Theorem 2.

Consider

$$\dot{x}(t) = f(x, t) \tag{1}$$

with $f: W \times \mathbb{R} \to \mathbb{R}^n$, W open, $W \subset \mathbb{R}^n$, $0 \in W$ and $f(0, t) = 0$ for all $t \in \mathbb{R}$. $f(x, t)$ is measurable in t for each fixed x. Furthermore, we assume that conditions are imposed on (1) such that the existence and uniqueness of its solutions is secured. These conditions will be a standard assumption on all the differential equations in the paper. Of these conditions, we single out the local Lipschitz condition. This local Lipschitz condition will be used in the course of the proofs of the theorems hereafter: f is locally Lipschitz on W, i.e., $\forall x \in W$, $\exists$ a neighborhood $\mathcal{N}(x) \subset W$, such that the restriction $f|_{\mathcal{N}(x)}$ is Lipschitz with Lipschitz function $l_x(t)$. We assume that $l_x(t)$ is bounded $\forall t \in \mathbb{R}$.

Theorem 1. *Consider a function $V: U \times \mathbb{R} \to \mathbb{R}$ with $U \subset W$ an open neighborhood of* 0. *We assume that the following conditions are satisfied*:

- Condition 1: *There exist positive numbers $v_{\min}$ and $v_{\max}$ such that for all $x \in U$, $v_{\min} x^T x = v_{\min}\|x\|^2 \le V(x,t) \le v_{\max} x^T x = v_{\max}\|x\|^2$. ($V(0,t) = 0$, $\forall t$.)*
- Condition 2: *There exists an increasing sequence of times t_k^* ($k \in \mathbb{Z}$) with $t_k^* \to \infty$ as $k \to \infty$ and $t_k^* \to -\infty$ as $k \to -\infty$, $\exists T > 0$, $t_{k+1}^* - t_k^* \le T$ ($\forall k \in \mathbb{Z}$), $\exists v > 0$ and an open set $U' \subset U$ containing the origin such that the Liapunov function $V(x,t)$ has the property that $\forall k \in \mathbb{Z}$ and $\forall x(t_k^*) \in U' \setminus \{0\}$,*

$$V(x(t_{k+1}^*), t_{k+1}^*) - V(x(t_k^*), t_k^*) \le -v\|x(t_k^*)\|^2 = -v x^T(t_k^*) x(t_k^*) < 0. \tag{2}$$

Here $x(t_{k+1}^*)$ is the solution of (1) *at t_{k+1}^* with initial condition $x(t_k^*)$ at t_k^*.*

Then the equilibrium point $x = 0$ of (1) *is exponentially stable.*

Proof. For a proof, the reader is referred to [2]. ∎

As mentioned in the Introduction, we do not really focus on systems arising as $\dot{x} = f(x(t), t)$, but we focus on systems arising as $\dot{x}(t) = f(x(t), t, \alpha t)$. More precisely, we consider the system

$$\dot{x}(t) = f(x, t, \alpha t) \tag{3}$$

with $f: W \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}^n$, W open, $W \subset \mathbb{R}^n$, $0 \in W$ and $\alpha \in \mathbb{R}_0^+$. Assume that $f(0, t, \alpha t) = 0$ for all $t \in \mathbb{R}$ and for all $\alpha \in \mathbb{R}_0^+$. We assume that f is measurable in the second and the third variable for each fixed $x \in W$ and that f is locally Lipschitz in W, i.e., $\forall x \in W$, $\exists$ neighborhood $\mathcal{N}(x) \subset W$ such that $f|_{\mathcal{N}(x)}$ is Lipschitz with Lipschitz functions $l_x(t, \alpha t)$. We assume that $l_x(t, \alpha t)$ is bounded $\forall t \in \mathbb{R}$ with a bound independent of α.

3. Properties

Before studying the partial averaging results, we first introduce a number of properties which will be useful in formulation of the averaging theorems.

Property 1. The system $\dot{x}(t) = \bar{f}(x(t), t)$ has an exponentially stable equilibrium point $x = 0$ with region of convergence $B_r(0) \subset W$ when there exist a $k > 1$ and a $\gamma > 0$ such that $B_{kr}(0) \subset W$ and for all $x_0 \in B_r(0)$, all $t_0 \in \mathbb{R}$ and all $t \ge t_0$,

$$\|\Phi(t, x_0, t_0)\| \le k\|x_0\| e^{-\gamma(t - t_0)}. \tag{4}$$

Here, $\Phi(t, x_0, t_0)$ denotes the solution of $\dot{x}(t) = \bar{f}(x, t)$ with initial condition $x_0 \in B_r(0)$ at t_0. When (4) is satisfied, $\Phi(t, x_0, t_0) \in B_{kr}(0)$ for all $x_0 \in B_r(0)$, for all t_0 and for all $t \ge t_0$.

Property 2. The systems $\dot{x}(t) = \bar{f}(x(t), t)$ and $\dot{x}(t) = f(x(t), t, \alpha t)$ satisfy Property 2 on $X \subset W$ when there exists an $\eta_1 > 0$ such that for all $\eta \ge \eta_1$, for all $x \in X$

and for all $\tau \in \mathbb{R}$,

$$\left\| \int_{\tau-1/\eta}^{\tau+1/\eta} (f(x,t,\eta^2 t) - \bar{f}(x,t))\, dt \right\| \leq N(\eta)\|x\|. \tag{5}$$

Here, $N(\cdot)$: $[0,+\infty) \to [0,+\infty)$ is a continuous function with $\lim_{\eta\to\infty} \eta N(\eta) = 0$.

Property 3. The system $\dot{x}(t) = \bar{f}(x(t),t)$ satisfies Property 3 on $B_r(0)$ when

- $\bar{f}(x,t)$ is continuous in t and three times continuously differentiable with respect to x for all $x \in B_r(0)$,
- there exist $M_1 > 0$, $M_2 > 0$ and $M_3 > 0$ such that $\forall x \in B_r(0)$, $\forall t \in \mathbb{R}$ and $\forall i,j,l,m \in \{1,\ldots,n\}$,

$$\begin{gathered} \left| \frac{\partial \bar{f}_i}{\partial x_j}(x,t) \right| \leq M_1, \\ \left| \frac{\partial^2 \bar{f}_i}{\partial x_j\, \partial x_l}(x,t) \right| \leq M_2 \quad \text{and} \quad \left| \frac{\partial^3 \bar{f}_i}{\partial x_j\, \partial x_j\, \partial x_m}(x,t) \right| \leq M_3. \end{gathered} \tag{6}$$

4. Partial Averaging: Part 1

In this section we study the exponential stability of differential equations of the form $\dot{x} = f(x,t,\alpha t)$. Theorem 2 requires the existence of a system $\dot{x} = \bar{f}(x,t)$ and a Liapunov function $V(x,t)$ satisfying appropriate conditions. Although Theorem 2 provides no tool for constructing an appropriate $\bar{f}(x,t)$ and an appropriate $V(x,t)$, Theorem 2 is a useful result. This is illustrated by means of a generalization of the circle criterion in Section 5. Knowledge of the system under investigation gives enough information to construct $\bar{f}(x,t)$ and $V(x,t)$. In other cases, however, it might be quite difficult to construct an $\bar{f}(x,t)$ and a $V(x,t)$. Therefore, the construction of $V(x,t)$ and $\bar{f}(x,t)$ is discussed in Theorems 3 and 4.

The proof of Theorem 2 relies on Theorem 1. With (8), guaranteeing that the derivative of $V(x,t)$ along the flow of $\dot{x} = \bar{f}(x,t)$ is negative definite, and (5), we prove that $V(x,t)$ satisfies the conditions of Theorem 1 along the flow of $\dot{x} = f(x,t,\alpha t)$ when α is sufficiently large. This implies exponential stability of $\dot{x} = f(x,t,\alpha t)$.

Consider the system (3) where $U \subset W$ is an open neighborhood of the origin.

Theorem 2. *If*:

- Condition 1: *There exists a system $\dot{x} = \bar{f}(x,t)$ $(x \in U, t \in \mathbb{R})$ such that $\dot{x} = \bar{f}(x,t)$ and $\dot{x} = f(x,t,\alpha t)$ satisfy Property 2 on U.*

  We also require that $\bar{f}(0,t) = 0$ for all $t \in \mathbb{R}$. Moreover, we require the existence of a $\bar{K} > 0$ such that $\bar{f}(x,t)$ is Lipschitz in U with Lipschitz constant $\bar{K}$.
- Condition 2: *$V(0,t) = 0$ for all $t \in \mathbb{R}$ and there exist positive numbers $v_{\min}$, $v_{\max}$ and w such that for all $x \in U$ and for all $t \in \mathbb{R}$,*

$$v_{\min} x^T x = v_{\min}\|x\|^2 \leq V(x,t) \leq v_{\max}\|x\|^2 = v_{\max} x^T x \tag{7}$$

and

$$\frac{\partial V}{\partial t}(x,t) + \frac{\partial V}{\partial x}(x,t)\bar{f}(x,t) \le -w\|x\|^2 = -wx^T x. \tag{8}$$

Moreover, for all $t \in \mathbb{R}$, $(\partial V/\partial x)(0,t) = 0$, *and* $(\partial V/\partial x)(x,t)$ *is Lipschitz on* U *with Lipschitz constant* K_v. *Moreover, for all* $t \in \mathbb{R}$, $(\partial^2 V/\partial x\,\partial t)(0,t) = 0$, *and* $(\partial^2 V/\partial x\,\partial t)(x,t)$ *is Lipschitz on* U *with Lipschitz constant* K_{vt}.

Then there exists an $\alpha_1 > 0$ *such that for all* $\alpha > \alpha_1$ *the equilibrium point* $x = 0$ *of the system* (3) *is exponentially stable.*

Proof. For the proof of the theorem see Appendix A. ■

Remark 1. When $f(x,t,\alpha t)$, $\bar{f}(x,t)$, $(\partial V/\partial x)(x,t)$ and $(\partial^2 V/\partial x\,\partial t)(x,t)$ are globally Lipschitz ($W = U = \mathbb{R}^n$) and all the conditions of Theorem 2 hold in $\mathbb{R}^n$, then the equilibrium point $x = 0$ of (3) is globally exponentially stable.

5. Generalization of the Circle Criterion

In this section we illustrate the use of Theorem 2 by means of a generalization of the circle criterion. For clarity, we first recall the most important topics of the circle criterion [9, pp. 399–419].

Consider the SISO linear time-invariant system

$$\dot{x}(t) = Ax(t) + bu(t), \qquad y(t) = cx(t), \tag{9}$$

where the matrix A is Hurwitz. The pair (A,b) is controllable and the pair (A,c) is observable. Consider the feedback

$$u(t) = -k(x(t),t)cx(t) \tag{10}$$

with $k(x,t)$ the gain of the feedback. We require that $k(x,t)cx$ is locally Lipschitz in x and piecewise continuous in t. The Lipschitz functions are bounded in time t. There exists an $a > 0$ and a $\bar{k} > 0$ such that for all $t \in \mathbb{R}$ and all x with $\|x\| < a$,

$$0 \le k(x,t) \le \bar{k}. \tag{11}$$

When there exist a symmetric positive definite matrix P, an n-dimensional vector L and a constant $\varepsilon > 0$ such that

$$A^T P + PA = -L^T L - \varepsilon P, \qquad Pb = c^T \bar{k} - \sqrt{2} L^T, \tag{12}$$

then the circle criterion [9, pp. 407–408] says that the equilibrium point $x = 0$ of the system

$$\dot{x}(t) = Ax(t) - bk(x(t),t)cx(t) \tag{13}$$

is exponentially stable.

The proof of this result relies on the classical Liapunov theorem for exponential stability. Indeed, based on (12) it is clear that for all x with $\|x\| < a$, the quadratic Liapunov function $V(x) = x^T Px$ has, along the solution (13), a derivative $\dot{V}(x,t) \le -\varepsilon x^T Px$. The classical theorem of Liapunov for exponential stability proves that the equilibrium point $x = 0$ of (13) is exponentially stable.

Remark 2. When (11) is valid for all $t \in \mathbb{R}$ and all $x \in \mathbb{R}^n$, then global exponential stability of the equilibrium point $x = 0$ of (13) follows.

5.1. Partial Averaging: the Circle Criterion

In this section we prove the exponential stability of systems described as a feedback interconnection of a fast time-varying linear SISO system

$$\dot{x}(t) = A(\alpha t)x(t) + bu(t), \qquad y(t) = cx(t) \tag{14}$$

and a feedback described by (10) and (11). The system matrix $A(\alpha t)$ is bounded and measurable. We assume that $k(x,t)cx$ is Lipschitz in $\{x \mid \|x\| < a\}$ and that $k(x,t)$ is piecewise continuous in t. We consider for each $\alpha > 0$ the system

$$\dot{x}(t) = A(\alpha t)x(t) - bk(x(t),t)cx(t). \tag{15}$$

We require the existence of a continuous nondecreasing function $M(\cdot)$: $[0,+\infty) \to [0,+\infty)$ where $\lim_{\sigma\to\infty} \sigma^{-1}M(\sigma) = 0$ such that for all $t_1, t_2 \in \mathbb{R}$ with $t_2 > t_1$,

$$\left\| \int_{t_1}^{t_2} (A(t) - \bar{A})\, dt \right\| \le M(t_2 - t_1) \qquad \text{with} \quad \bar{A} := \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{+T} A(t)\, dt. \tag{16}$$

We assume that $\bar{A}$ is Hurwitz. The pair $(\bar{A}, b)$ is controllable and the pair $(\bar{A}, c)$ is observable. When there exist a positive definite matrix P, an n-dimensional L and a constant $\varepsilon > 0$ such that

$$\bar{A}^T P + P\bar{A} = -L^T L - \varepsilon P \quad \text{and} \quad Pb = c^T \bar{k} - \sqrt{2} L^T, \tag{17}$$

then the circle criterion proves exponential stability of the equilibrium point $x = 0$ of the system

$$\dot{x}(t) = \bar{A}x(t) - bk(x(t),t)cx(t). \tag{18}$$

By setting $f(x,t,\alpha t) = A(\alpha t)x - bk(x,t)cx$ and $\bar{f}(x,t) = \bar{A}x - bk(x,t)cx$, we use Theorem 2. Since for all $\tau \in \mathbb{R}$, all $\eta > 0$ and all x with $\|x\| < a$,

$$\left\| \int_{\tau-1/\eta}^{\tau+1/\eta} f(x,t,\eta^2 t) - \bar{f}(x,t)\, dt \right\| = \left\| \int_{\tau-1/\eta}^{\tau+1/\eta} (A(\eta^2 t) - \bar{A})x\, dt \right\| \tag{19}$$

$$= \left\| \frac{1}{\eta^2} \int_{\tau\eta^2-\eta}^{\tau\eta^2+\eta} (A(t) - \bar{A})x\, dt \right\| \le \frac{M(2\eta)}{\eta^2} \|x\|, \tag{20}$$

it is clear that (5) of Property 2 is satisfied with $N(\eta) = M(2\eta)/\eta^2$ and $U = \{x \mid \|x\| < a\}$.

Relying on the Lipschitz property of $k(x,t)cx$ on $\{x \mid \|x\| < a\}$, the Lipschitz property of $\bar{f}(x,t)$—required by Condition 1 of Theorem 2—is satisfied. Relying on (17), $V(x) = x^T Px$ has, along the solution of (18), a derivative

$$\dot{V}(x,t) \le -\varepsilon x^T Px \le -\varepsilon \lambda_{\min(P)} \|x\|^2 \tag{21}$$

for all x with $\|x\| < a$. Relying on (21), (8) in Condition 2 of Theorem 2 is satisfied with $w = \varepsilon\lambda_{\min}(P)$ and $U = \{x \mid \|x\| < a\}$. Since $V(x) = x^T Px$ is a quadratic Liapunov function, (7) and the smoothness conditions on $V(x,t)$ in Condition 2 of Theorem 2 are satisfied.

Relying on Theorem 2, (17) implies the existence of an $\alpha_1 > 0$ such that for all $\alpha > \alpha_1$ the equilibrium point $x = 0$ of (15) is exponentially stable.

Remark 3. When (11) is valid for all $t \in \mathbb{R}$ and all $x \in \mathbb{R}^n$—and when $k(x,t)cx$ is globally Lipschitz—then (17) implies the existence of an $\alpha_1 > 0$ such that for all $\alpha > \alpha_1$ the equilibrium point $x = 0$ of (15) is globally exponentially stable. This global result is obtained by using Theorem 2 and Remark 1.

Remark 4. In this section we generalized the circle criterion by allowing a time-varying (fast time-varying) matrix $A(\alpha t)$. Instead of allowing a time-varying $A(\alpha t)$, it is also possible to allow a fast time-varying $b(\alpha t)$ or a fast time-varying $c(\alpha t)$. Another generalization of the circle criterion allows us to replace $k(x,t)$ in (13) by a partially fast time-varying $k(x,t,\alpha t)$. We chose not to develop these generalizations since they are quite similar to the development discussed in this section.

6. Partial Averaging: Part 2

By Theorem 2 it is possible to study the exponential stability of systems of the form $\dot{x}(t) = f(x,t,\alpha t)$. The use of Theorem 2 itself has been illustrated in the previous section. Notice however that Theorem 2 provides no tool for constructing $\bar{f}(x,t)$ and $V(x,t)$ which have to satisfy Conditions 1 and 2 of Theorem 2. By (8), $\dot{x} = \bar{f}(x,t)$ and $V(x,t)$ are related to each other. For a system $\dot{x} = \bar{f}(x,t)$ satisfying (5) and having an exponentially stable null solution, the converse theorem of Liapunov for exponential stability [9] guarantees the existence of a $V(x,t)$ satisfying (7) and (8). This leads to Theorem 3, i.e., we show that exponential stability of $\dot{x}(t) = \bar{f}(x,t)$ implies exponential stability of the original system $\dot{x}(t) = f(x,t,\alpha t)$ with α large enough. Theorem 3 provides no clue for constructing $\bar{f}(x,t)$. The construction of $\bar{f}(x,t)$ is discussed in Theorem 4.

Consider for all $\alpha > 0$ the system (3).

Theorem 3. *If*:

- Condition 1: *There exists a system*

$$\dot{x}(t) = \bar{f}(x,t) \tag{22}$$

with $\bar{f}\colon W \times \mathbb{R} \to \mathbb{R}^n$ *and* $\bar{f}(0,t) = 0$ *for all* $t \in \mathbb{R}$. *Assume there exists a* $\bar{K} > 0$ *such that* $\bar{f}(x,t)$ *is Lipschitz in* W *with Lipschitz constant* $\bar{K}$.

There exists an $r_0 > 0$ *such that the system* (22) *has an exponentially stable equilibrium point* $x = 0$ *with region of convergence* $B_{r_0}(0)$ (*see Property* 1 *in Section* 3). *We additionally require that* $B_{kr_0}(0) \subset W$.

- Condition 2: *The systems* $\dot{x}(t) = \bar{f}(x, t)$ *and* $\dot{x}(t) = f(x, t, \alpha t)$ *satisfy Property* 2 *on* $B_{r_0}(0)$.
- Condition 3: *The system* $\dot{x}(t) = \bar{f}(x, t)$ *satisfies Property* 3 *on* $B_{kr_0}(0)$.

Then there exists an $\alpha_1 > 0$ *such that for all* $\alpha > \alpha_1$ *the equilibrium point* $x = 0$ *of the system* (3) *is locally exponentially stable.*

Proof. For the proof of this theorem see Appendix B. ■

Remark 5. Even when $f(x, t, \alpha t)$ and $\bar{f}(x, t)$ are globally Lipschitz ($W = \mathbb{R}^n$ and $r_0 = \infty$) and the conditions of Theorem 3 hold in $\mathbb{R}^n$, Theorem 3 does *not* prove the existence of an $\alpha_1 > 0$ such that for all $\alpha > \alpha_1$ the system $\dot{x} = f(x, t, \alpha t)$ is globally exponentially stable. Indeed, Theorem 3 relies on Theorem 2. In order to obain a global exponential stability result, Theorem 2 and Remark 1 do not only require that the conditions of Theorem 2 are satisfied in $\mathbb{R}^n$, but they also require that $(\partial V/\partial x)(x, t)$ and $(\partial^2 V/\partial x\, \partial t)(x, t)$ are globally Lipschitz. These global Lipschitz properties of $(\partial V/\partial x)(x, t)$ and $(\partial^2 V/\partial x\, \partial t)(x, t)$ are not guaranteed since the right-hand side of (85) and the right-hand side of (92) tend to infinity as $\|x\|$ goes to infinity. Therefore, Theorem 3 proves local partial averaging results. Only when additional conditions are imposed, is it possible to obtain a global stability result.

Remark 6. When $f(x, t, \alpha t)$ and $\bar{f}(x, t)$ are globally Lipschitz and condition (6) (see Condition 3) is replaced by the condition that for all $x \in \mathbb{R}^n$, all $t \in \mathbb{R}$ and all $i, j, l, m \in \{1, \ldots, n\}$,

$$\begin{gathered} \left| \frac{\partial \bar{f}_i}{\partial x_j}(x, t) \right| \leq M_1, \\ \left| \frac{\partial^2 \bar{f}_i}{\partial x_j\, \partial x_l}(x, t) \right| = M_2 = 0 \quad \text{and} \quad \left| \frac{\partial^3 \bar{f}_i}{\partial x_j\, \partial x_j\, \partial x_m}(x, t) \right| = M_3 = 0, \end{gathered} \tag{23}$$

then there exists an $\alpha_1 > 0$ such that for all $\alpha > \alpha_1$ the equilibrium point $x = 0$ of (3) is globally exponentially stable. Indeed, when (23) is satisfied, $M_5(T_V) = 0$. This implies—relying on (85) and (127)—that $M_9(\|x\|) = M_9$ is independent of $\|x\|$. Relying on (117), $M_7(T', \|x\|) = M_7(T')$ is independent of $\|x\|$. Relying on (23) and (123), $M_8(T_V, \|x\|) = 0$ for all $\|x\|$. Since $M_5(T_V) = 0$ and $M_8(T_V, \|x\|) = 0$, (92) implies that $M_{10}(\|x\|) = M_{10}$ is independent of $\|x\|$. This implies that for all $t \in \mathbb{R}$ and all $x \in \mathbb{R}^n$,

$$\sum_{i,j=1}^{n} |J_{1ij}(x, t)| \leq M_9 \quad \text{and} \quad \sum_{i,j=1}^{n} |J_{2ij}(x, t)| \leq M_{10}. \tag{24}$$

This implies that $(\partial^2 V/\partial x\, \partial t)(x, t)$ and $(\partial V/\partial x)(x, t)$ are globally Lipschitz. Therefore, there exists an $\alpha_1 > 0$ such that for all $\alpha > \alpha_1$ the equilibrium point $x = 0$ of (3) is globally exponentially stable. For linear systems, $M_2 = M_3 = 0$ such that Theorem 3 and the present remark indeed prove *global* exponential stability of the equilibrium point $x = 0$.

7. Partial Averaging: Part 3

Theorems 2 and 3 provide no method to construct a system $\dot{x}(t) = \bar{f}(x, t)$ satisfying the appropriate conditions. In this section we prove Theorem 4. By calculating the partially averaged system $\dot{x}(t) = \bar{f}(x(t), t)$ as

$$\bar{f}(x, \tau) := \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{+T} f(x, \tau, t)\, dt \tag{25}$$

for all $x \in W$ and for all $\tau \in \mathbb{R}$,[1] Theorem 4 formulates the partial averaging result in a way closely related to the classical averaging results [2], [9].

Consider for all $\alpha > 0$ the system (3).

Theorem 4. *If*:

- Condition 1: *There exists an* $r_0 > 0$ *such that the partially averaged system* $\dot{x} = \bar{f}(x, t)$ *of* $\dot{x} = f(x, t, \alpha t)$ *defined by* (25) *has an exponentially stable equilibrium point* $x = 0$ *with region of convergence* $B_{r_0}(0)$ (*Property* 1). *It is also required that* $B_{kr_0}(0) \subset W$.
- Condition 2: *There exists a continuous nondecreasing function* $M(\cdot)$: $[0, +\infty) \to [0, +\infty)$ *with* $\lim_{\sigma\to\infty} \sigma^{-1} M(\sigma) = 0$ *such that for all* $x \in B_{r_0}(0)$, *all* $\tau \in \mathbb{R}$ *and all* $t_1, t_2 \in \mathbb{R}$ *with* $t_2 > t_1$,

$$\left\| \int_{t_1}^{t_2} (f(x, \tau, t) - \bar{f}(x, \tau))\, dt \right\| \le M(t_2 - t_1)\|x\|. \tag{26}$$

- Condition 3: *There exists* $l \in \mathbb{R}_0^+$ *and a number* $n_d \in \mathbb{N}$ *such that in every open interval of length* l, $f(x, \tau, t) - \bar{f}(x, \tau)$ *has maximum* n_d *discontinuities in* τ, *when* x *and* t *are fixed.*

  We require the existence of a continuous nondecreasing function $K_{r_0}(\cdot)$: $[0, +\infty) \to [0, +\infty)$ *with* $K_{r_0}(0) = 0$ *such that for all* $\tau_1, \tau_2 \in \mathbb{R}$ *with* $\tau_2 > \tau_1$, *all* $t_1 \in \mathbb{R}$ *and all* $x \in B_{r_0}(0)$,

$$\|(f(x, \tau_2, t_1) - \bar{f}(x, \tau_2)) - (f(x, \tau_1, t_1) - \bar{f}(x, \tau_1))\| \le K_{r_0}(\tau_2 - \tau_1)\|x\| \tag{27}$$

  when the interval $[\tau_1, \tau_2]$ *contains no discontinuities.*
- Condition 4: *The partially averaged system* (25) *satisfies Property* 3 *on* $B_{kr_0}(0)$.

Then there exists an $\alpha_1 > 0$ *such that for all* $\alpha > \alpha_1$ *the equilibrium point* $x = 0$ *of* (3) *is locally exponentially stable.*

Proof. For the proof of this theorem see Appendix C. ■

Remark 7. When the conditions of Theorem 4 are satisfied globally ($W = \mathbb{R}^n$ and $r_0 = \infty$), then the conditions of Theorem 3 are also satisfied globally. As explained by Remark 5, this does *not* imply the existence of an $\alpha_1 > 0$ such that for all $\alpha > \alpha_1$ the system $\dot{x} = f(x, t, \alpha t)$ is globally exponentially stable.

---

[1] $\bar{f}$: $W \times \mathbb{R} \to \mathbb{R}^n$, $\bar{f}(0, t) = 0$ for all $t \in \mathbb{R}$. In the proof of Theorem 4 we require the existence of a $\bar{K} > 0$ such that $\bar{f}(x, t)$ is Lipschitz on W with Lipschitz constant $\bar{K}$.

Remark 8. When $f(x, t, \alpha t)$ and $\bar{f}(x, t)$ are globally Lipschitz ($W = \mathbb{R}^n$), when condition (6) is replaced by the condition that for all $x \in \mathbb{R}^n$, all $t \in \mathbb{R}$ and all $i, j, l, m \in \{1, \dots, n\}$,

$$\begin{gathered} \left| \frac{\partial \bar{f}_i}{\partial x_j}(x, t) \right| \le M_1, \\ \left| \frac{\partial^2 \bar{f}_i}{\partial x_j\, \partial x_l}(x, t) \right| = M_2 = 0 \quad \text{and} \quad \left| \frac{\partial^3 \bar{f}_i}{\partial x_j\, \partial x_j\, \partial x_m}(x, t) \right| = M_3 = 0, \end{gathered} \tag{28}$$

and when the conditions of Theorem 4 are satisfied globally, then Theorem 3 and Remark 6 prove the existence of an $\alpha_1 > 0$ such that for all $\alpha > \alpha_1$ the equilibrium point $x = 0$ of (3) is globally exponentially stable.

Remark 9. Recall that Theorem 4 relies on Theorems 2 and 3 and therefore (25) provides a system $\dot{x} = \bar{f}(x, t)$ satisfying (5). We show by contradiction that $\dot{x}(t) = \bar{f}(x(t), t)$ is the only time-continuous system satisfying (5). Suppose there are two functions $\bar{f}_1(x, t)$ and $\bar{f}_2(x, t)$, continuous in t, which satisfy (5). By Condition 1 of Theorem 2 and Property 2, one obtains that for all $\tau \in \mathbb{R}$,

$$\begin{gathered} \lim_{\eta \to \infty} \frac{\eta}{2} \int_{\tau - 1/\eta}^{\tau + 1/\eta} (f(x, t, \eta^2 t) - \bar{f}_1(x, t))\, dt = 0 \quad \text{and} \\ \lim_{\eta \to \infty} \frac{\eta}{2} \int_{\tau - 1/\eta}^{\tau + 1/\eta} (f(x, t, \eta^2 t) - \bar{f}_2(x, t)\, dt = 0. \end{gathered} \tag{29}$$

This implies that for all $\tau \in \mathbb{R}$,

$$\lim_{\eta \to \infty} \frac{\eta}{2} \int_{\tau - 1/\eta}^{\tau + 1/\eta} (\bar{f}_1(x, t) - \bar{f}_2(x, t))\, dt = 0. \tag{30}$$

In turn, this implies that for all $\tau \in \mathbb{R}, \bar{f}_1(x, \tau) = \bar{f}_2(x, \tau)$.

8. Partial Averaging: Applications

In this section we illustrate the use of Theorem 4 by means of two applications. The first application, which arises in system identification, deals with a linear system. The second application deals with a specific nonlinear example.

The advantage of using Theorem 4 is that (25) can be used to calculate the partially averaged system. In order to prove exponential stability of $\dot{x} = f(x, t, \alpha t)$, it is sufficient to prove exponential stability of $\dot{x} = \bar{f}(x, t)$ (and to verify the more technical conditions of Theorem 4). On the other hand, Theorem 4 imposes more smoothness conditions on $\dot{x} = \bar{f}(x, t)$ than Theorem 2 does.

8.1. Partial Averaging of Linear Systems

Systems of the form $\dot{x}(t) = -m(t)m^T(t)x(t)$, arising in system identification, have been extensively studied in [3]–[5]. Theorem 4 may be used to extend these results to systems arising as $\dot{x}(t) = f(\alpha t)m(t)m^T(t)x(t)$ where $f\colon \mathbb{R} \to \mathbb{R}$ attains positive and negative values.

Consider the system

$$\dot{x}(t) = f(\alpha t)m(t)m^T(t)x(t) \tag{31}$$

with $m\colon \mathbb{R} \to \mathbb{R}^n$ and $f\colon \mathbb{R} \to \mathbb{R}$ attaining positive and negative values ($\alpha \in \mathbb{R}_0^+$). Without loss of generality we assume that for all $t \in \mathbb{R}$, $\|m(t)\| = 1$. We require that $m(t)$ is differentiable and that $f(t)$ is measurable. Moreover, $\|\dot{m}(t)\|$ and $|f(t)|$ are assumed to be bounded, i.e., there exist $\dot{m}_M > 0$ and $f_M > 0$ such that, for all $t \in \mathbb{R}$, $\|\dot{m}(t)\| \leq \dot{m}_M$ and $|f(t)| \leq f_M$.

Illustration 1. *If*:

- Condition 1:

$$\bar{f} := \lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{+T} f(t)\,dt < 0. \tag{32}$$

- Condition 2: *There exists a continuous nondecreasing function* $M(\cdot)$: $[0,+\infty) \to [0,+\infty)$ *with* $\lim_{\sigma\to\infty} \sigma^{-1}M(\sigma) = 0$ *such that for all* $t_1, t_2 \in \mathbb{R}$ *with* $t_2 > t_1$,

$$\left|\int_{t_1}^{t_2} (f(t) - \bar{f})\,dt\right| \leq M(t_2 - t_1). \tag{33}$$

- Condition 3: *There exists a* $T > 0$ *such that for all* $t \in \mathbb{R}$,

$$\alpha' I \leq \int_t^{t+T} m(\tau)m^T(\tau)\,d\tau \leq \beta' I \tag{34}$$

  with $\beta' \geq \alpha' > 0$.

Then there exists an $\alpha_1 > 0$ *such that for all* $\alpha > \alpha_1$, *the system* (31) *has an exponenentially stable equilibrium point* $x = 0$.

Proof. The application is implied by Theorem 4 with $f(x,t,\alpha t) = f(\alpha t)m^T(t)m(t)x$ and $\bar{f}(x,t) = \bar{f}m^T(t)m(t)x$. Conditions 1 and 3 guarantee by [5] exponential stability of the partially averaged system

$$\dot{x}(t) = \bar{f}m(t)m^T(t)x(t). \tag{35}$$

Since for all $t_1, t_2 \in \mathbb{R}$ $(t_2 > t_1)$,

$$\int_{t_1}^{t_2} f(x,\tau,t) - \bar{f}(x,\tau)\,dt = \int_{t_1}^{t_2} (f(t) - \bar{f})\,dt\; m^T(\tau)m(\tau)x \tag{36}$$

and since $\|m(\tau)\| = 1$,

$$\left\|\int_{t_1}^{t_2} f(x,\tau,t) - \bar{f}(x,\tau)\,dt\right\| \leq M(t_2 - t_1)\|x\| \tag{37}$$

such that (26) is satisfied.

Since for all $\tau_1, \tau_2 \in \mathbb{R}$ $(\tau_2 > \tau_1)$ and for all t_1,

$$\begin{aligned} &\|(f(x, \tau_2, t_1) - \bar{f}(x, \tau_2)) - (f(x, \tau_1, t_1) + \bar{f}(x, \tau_1))\| \\ &\quad = \|(f(t_1) - \bar{f})(m^T(\tau_2)m(\tau_2) - m^T(\tau_1)m(\tau_1))x\| \\ &\quad \leq 2f_M\|(m^T(\tau_2) - m^T(\tau_1))(m(\tau_2) + m(\tau_1))\| \, \|x\| \\ &\quad \leq 4f_M\dot{m}_M(\tau_2 - \tau_1)\|x\|, \end{aligned} \tag{38}$$

Condition 3 of Theorem 4 is also satisfied with $K_{r_0}(\tau_2 - \tau_1) = 4f_M\dot{m}_M(\tau_2 - \tau_1)$.

By applying Theorem 4, there exists an $\alpha_1 > 0$ such that for all $\alpha > \alpha_1$, (31) has an exponentially stable equilibrium point $x = 0$. ■

8.2. Partial Averaging of Nonlinear Systems

Consider the scalar system

$$\dot{x}(t) = f(x(t), t, \alpha t) = -x(t)(2 + \sin(t + \cos(x(t)^2)))(1 + 10\sin(\alpha t)) \tag{39}$$

which is defined for all $x \in \mathbb{R}$ and for all $t \in \mathbb{R}$. By (25), the partially averaged system $\dot{x}(t) = \bar{f}(x(t), t) = -x(t)(2 + \sin(t + \cos(x(t)^2)))$. This system is globally exponentially stable. Indeed, with $V(x, t) = V(x) = x^2$, the derivative of $V(x)$ along the trajectories of the partially averaged system equals $-2x^2(2 + \sin(t + \cos(x^2))) \leq -2x^2$.

It is a straightforward calculation to verify that Conditions 2 and 3 of Theorem 4 are satisfied for all $x \in \mathbb{R}$. Condition 4 is also satisfied but there exist no M_1, M_2 and M_3 such that (6) is satisfied for all $x \in \mathbb{R}$. It is straightforward to verify that for every finite (possibly large) r_0, there exist an $M_1(r_0)$, $M_2(r_0)$ and $M_3(r_0)$ such that (6) is satisfied. By Theorem 4 we obtain that for every r_0, there exists a $\alpha_1(r_0)$ such that for all $\alpha > \alpha(r_0)$ the equilibrium point $x = 0$ is locally exponentially stable. The region of convergence will depend on r_0.

9. Counterexamples

In this section we illustrate the use of some conditions required by Theorems 2–4. We do this by considering some systems which violate one of the conditions required by the theorems and showing that exponential stability of the (partially) averaged system does not imply exponential stability of the original fast time-varying systems.

9.1. First Counterexample

Consider the function $s(\cdot)$: $\mathbb{R} \to \mathbb{R}$ where $s(\tau) = +1$ when $\tau \geq 0$ and $s(\tau) = -3$ when $\tau < 0$. The scalar system $\dot{x}(t) = s(\alpha t)x(t)$ is unstable for all $\alpha > 0$. By calculating the averaged system by (25), we obtain the exponentially stable system $\dot{x} = -x$. Although the (partially) averaged system is exponentially stable, the original system is unstable, even for very large α. The reason is the violation of Condition

2 of Theorem 4. By taking $t_1 = 0$,

$$\left\| \int_{t_1}^{t_2} (f(x,\tau,t) - \bar{f}(x,\tau))\, dt \right\| = \left\| \int_0^{t_2} 2x\, dt \right\| = 2xt_2. \tag{40}$$

In order to satisfy (26), $M(t_2) \geq 2t_2$ which violates the condition that $\lim_{\sigma\to\infty} \sigma^{-1} M(\sigma) = 0$.

9.2. Second Counterexample

Consider for all $\alpha > 0$ and for all $\beta > 0$ the scalar system

$$\dot{x} = f_1(\alpha t) f_2(\beta t) x. \tag{41}$$

The functions $f_1(\cdot)\colon \mathbb{R} \to \mathbb{R}$ and $f_2(\cdot)\colon \mathbb{R} \to \mathbb{R}$ are periodic functions with period 2π. When $t \in [0,\pi[$ then $f_1(t) = \sin t$ and $f_2(t) = 0$. When $t \in [\pi, 2\pi[$ then $f_1(t) = f_2(t) = \sin t$.

For each fixed $\beta > 0$, the partially averaged system

$$\dot{x} = \frac{1}{\pi} f_2(\beta t) x \tag{42}$$

is exponentially stable. By Theorem 4 there exists an $\alpha_1(\beta) > 0$ such that for all $\alpha > \alpha_1(\beta)$, (41) is exponentially stable. Since (41) is unstable when $\alpha = \beta$, we know that $\alpha_1(\beta)$ indeed depends on β and that $\alpha > \alpha_1(\beta) > \beta$. This illustrates it is only allowed to average f_1 when f_1 is fast time-varying with respect to f_2. In the system

$$\dot{x} = f_1(\alpha t) f_2(\alpha t) x \tag{43}$$

neither f_1 nor f_2 can be averaged separately. The system (43) is fast time-varying and the averaged system equals

$$\dot{x} = \overline{f_1 f_2} x \qquad \text{where} \quad \overline{f_1 f_2} := \frac{1}{2\pi} \int_0^{2\pi} f_1(t) f_2(t)\, dt = \tfrac{1}{4}. \tag{44}$$

This example illustrates that exponential stability of $\dot{x} = \bar{f}(x,t)$ implies exponential stability of $\dot{x} = f(x,t,\alpha t)$ for α sufficiently large. It does not imply exponential stability of $\dot{x} = f(x,\alpha t,\alpha t)$.

Appendix A. Proof of Theorem 2

(**I**) Consider the Liapunov function $V(x,t)$. The derivative of this Liapunov function along the solutions of (3) with $\alpha = \eta^2$ is for all $x \in U$ given by

$$\dot{V}(x,t) = \frac{\partial V}{\partial t}(x,t) + \frac{\partial V}{\partial x}(x,t) f(x,t,\eta^2 t). \tag{45}$$

Take a $\tau \in \mathbb{R}$ and then for all $\eta > 0$ define

$$\Delta V\left(\tau + \frac{1}{\eta}, \tau - \frac{1}{\eta}\right) := V\left(x\left(\tau + \frac{1}{\eta}\right), \tau + \frac{1}{\eta}\right) - V\left(x\left(\tau - \frac{1}{\eta}\right), \tau - \frac{1}{\eta}\right). \tag{46}$$

Here $x(\tau + 1/\eta)$ is the solution of (3) evaluated at $\tau + 1/\eta$ with initial condition $x(\tau - 1/\eta)$ at $\tau - 1/\eta$. When $x(t) \in U$ for all $t \in [\tau - 1/\eta, \tau + 1/\eta]$, then by (45) and (46),

$$\Delta V\left(\tau + \frac{1}{\eta}, \tau - \frac{1}{\eta}\right) = \int_{\tau-1/\eta}^{\tau+1/\eta} \dot{V}(x(t), t)\, dt$$

$$= \int_{\tau-1/\eta}^{\tau+1/\eta} \frac{\partial V}{\partial t}(x(t), t) + \frac{\partial V}{\partial x}(x(t), t) f(x(t), t, \eta^2 t)\, dt. \quad (47)$$

Here $x(t)$ is the solution of (3) with $\alpha = \eta^2$ evaluated at t with initial condition $x(\tau - 1/\eta)$ at $\tau - 1/\eta$.

$$\Delta V\left(\tau + \frac{1}{\eta}, \tau - \frac{1}{\eta}\right) \quad (48)$$

$$= \int_{\tau-1/\eta}^{\tau+1/\eta} \frac{\partial V}{\partial t}\left(x\left(\tau - \frac{1}{\eta}\right), t\right)$$

$$+ \frac{\partial V}{\partial x}\left(x\left(\tau - \frac{1}{\eta}\right), t\right) f\left(x\left(\tau - \frac{1}{\eta}\right), t, \eta^2 t\right) dt \quad (49)$$

$$+ \int_{\tau-1/\eta}^{\tau+1/\eta} \left(\frac{\partial V}{\partial x}(x(t), t) f(x(t), t, \eta^2 t)\right.$$

$$\left. - \frac{\partial V}{\partial x}\left(x\left(\tau - \frac{1}{\eta}\right), t\right) f\left(x\left(\tau - \frac{1}{\eta}\right), t, \eta^2 t\right)\right) dt \quad (50)$$

$$+ \int_{\tau-1/\eta}^{\tau+1/n} \left(\frac{\partial V}{\partial t}(x(t), t) - \frac{\partial V}{\partial t}\left(x\left(\tau - \frac{1}{\eta}\right), t\right)\right) dt. \quad (51)$$

In order to evaluate (48), we need upper bounds for the absolute values of (50) and (51).

(**II**) Consider a closed ball $\bar{B}_\xi(0)$ centered at 0 and radius ξ small enough such that $\bar{B}_\xi(0) \subset U$. Since $\bar{B}_\xi(0)$ is compact, it can be covered with a finite number of open neighborhoods $\mathcal{N}(x) \subset W$, $x \in \bar{B}_\xi(0)$. Let K be the maximum of the upper bounds of the Lipschitz function $l_x(t, \alpha t)$ such that $\forall x, y \in \bar{B}_\xi(0)$, $\forall t \in \mathbb{R}$ and $\forall \eta > 0$,

$$\|f(x, t, \eta^2 t)\| \le K\|x\| \quad \text{and} \quad \|f(x, t, \eta^2 t) - f(y, t, \eta^2 t)\| \le K\|x - y\|. \quad (52)$$

Define $\xi' := \xi e^{-KT}$ with $T = 2/\eta$. Let $B_{\xi'}(0)$ be the open ball with center 0 and radius ξ'. By Lemma 1 and Remark 1 in [1] with $\mu = \xi$ and $\mu' = \xi'$, one obtains that for $t \in [\tau - 1/\eta, \tau + 1/\eta]$, $x(t) \in B_\xi(0) \subset U$ when $x(\tau - 1/\eta) \in B_{\xi'}(0)$. Therefore

$$x(t) = x\left(\tau - \frac{1}{\eta}\right) + \int_{\tau-1/\eta}^{t} f(x(s), s, \eta^2 s)\, ds \quad (53)$$

and

$$\|x(t)\| \leq \left\| x\left(\tau - \frac{1}{\eta}\right) \right\| + \int_{\tau-1/\eta}^{t} K\|x(s)\|\, ds. \tag{54}$$

By the Gronwall–Bellman lemma, one obtains for all $t \in [\tau - 1/\eta, \tau + 1/\eta]$ that

$$\|x(t)\| \leq \left\| x\left(\tau - \frac{1}{\eta}\right) \right\| e^{K(t-\tau+1/\eta)} \leq \left\| x\left(\tau - \frac{1}{\eta}\right) \right\| e^{2K/\eta} \tag{55}$$

when $x(\tau - 1/\eta) \in B_{\xi'}(0)$. One obtains by (53) that for all $t \in [\tau - 1/\eta, \tau + 1/\eta]$,

$$\left\| x(t) - x\left(\tau - \frac{1}{\eta}\right) \right\| \leq \frac{2K}{\eta} \left\| x\left(\tau - \frac{1}{\eta}\right) \right\| e^{2K/\eta}. \tag{56}$$

(**III**) Since $(\partial V/\partial x)(x,t)$ is Lipschitz on U, one obtains that $\forall x, y \in \bar{B}_{\xi}(0)$ and $\forall t \in \mathbb{R}$,

$$\left\| \frac{\partial V}{\partial x}(x,t) \right\| \leq K_v \|x\| \quad \text{and} \quad \left\| \frac{\partial V}{\partial x}(x,t) - \frac{\partial V}{\partial x}(y,t) \right\| \leq K_v \|x - y\|. \tag{57}$$

By defining $L_f V(x,t,\eta^2 t) := (\partial V/\partial x)(x,t) f(x,t,\eta^2 t)$, (50) is equal to

$$\int_{\tau-1/\eta}^{\tau+1/n} \left(L_f V(x(t), t, \eta^2 t) - L_f V\left(x\left(\tau - \frac{1}{\eta}\right), t, \eta^2 t \right) \right) dt. \tag{58}$$

When $x, y \in \bar{B}_{\xi}(0)$ then $\forall t \in \mathbb{R}$ and $\forall \eta > 0$,

$$\begin{aligned} &L_f V(x,t,\eta^2 t) - L_f V(y,t,\eta^2 t) \\ &\quad = \frac{\partial V}{\partial x}(x,t)(f(x,t,\eta^2 t) - f(x,t,\eta^2 t)) \\ &\quad\quad + \left(\frac{\partial V}{\partial x}(x,t) - \frac{\partial V}{\partial x}(y,t) \right) f(y,t,\eta^2 t) \end{aligned} \tag{59}$$

such that

$$\begin{aligned} |L_f V(x,t,\eta^2 t) - L_f V(y,t,\eta^2 t)| &\leq K_v \|x\| K \|x - y\| + K\|y\| K_v \|x - y\| \\ &\leq 2KK_v \|x - y\| \max\{\|x\|, \|y\|\}. \end{aligned} \tag{60}$$

By (55), (56) and (60),

$$\left| L_f V(x(t), t, \eta^2 t) - L_f V\left(x\left(\tau - \frac{1}{\eta}\right), t, \eta^2 t \right) \right| \leq \frac{4K^2 K_v}{\eta} e^{4K/\eta} \left\| x\left(\tau - \frac{1}{\eta}\right) \right\|^2. \tag{61}$$

We are set to evaluate upper bounds for the absolute values of (50). If $x(\tau - 1/\eta) \in B_{\xi'}(0)$, then by (61), (50) has an absolute value which is less than or equal to

$$2K_v K^2 e^{4K/\eta} \left(\frac{2}{\eta}\right)^2 \left\| x\left(\tau - \frac{1}{\eta}\right) \right\|^2. \tag{62}$$

(**IV**) We evaluate an upper bound for the norm of (51). Since $(\partial^2 V/\partial x\,\partial t)(x,t)$ is Lipschitz with Lipschitz constant K_{vt}, one obtains that $\forall x, y \in \bar{B}_\xi(0)$ and $\forall t \in \mathbb{R}$,

$$\left\|\frac{\partial^2 V}{\partial x\,\partial t}(x,t) - \frac{\partial^2 V}{\partial x\,\partial t}(y,t)\right\| \le K_{vt}\|x-y\| \quad \text{and} \quad \left\|\frac{\partial^2 V}{\partial x\,\partial t}(x,t)\right\| \le K_{vt}\|x\|. \tag{63}$$

Since $(\partial^2 V/\partial x\,\partial t)(x,t)$ is Lipschitz on U, $(\partial^2 V/\partial x\,\partial t)(x,t)$ and $\|(\partial^2 V/\partial x\,\partial t)(x,t)\|$ are continuous in x. If $x, y \in \bar{B}_\xi(0) \subset U$, then by the mean-value theorem [6, Theorem 8.4, p. 254, and Theorem 8.5, p. 259], there is a point z on the line segment l_{xy} joining x and y such that

$$\left|\frac{\partial V}{\partial t}(x,t) - \frac{\partial V}{\partial t}(y,t)\right| = \left|\frac{\partial^2 V}{\partial x\,\partial t}(z,t)(x-y)\right|. \tag{64}$$

By the convexity of $B_\xi(0)$, $l_{xy} \subset B_\xi(0)$ implying that for all $z \in l_{xy}$,

$$\left\|\frac{\partial^2 V}{\partial x\,\partial t}(z,t)\right\| \le K_{vt}\|z\| \le K_{vt}\max\{\|x\|, \|y\|\}. \tag{65}$$

By (56) and (64),

$$\begin{aligned}
&\left|\frac{\partial V}{\partial t}(x(t),t) - \frac{\partial V}{\partial t}\left(x\left(\tau-\frac{1}{\eta}\right),t\right)\right| \\
&\quad \le K_{vt}e^{2K/\eta}\left\|x\left(\tau-\frac{1}{\eta}\right)\right\|\,\left\|x(t) - x\left(\tau-\frac{1}{\eta}\right)\right\| \qquad (66)\\
&\quad \le \frac{2KK_{vt}}{\eta}e^{4K/\eta}\left\|x\left(\tau-\frac{1}{\eta}\right)\right\|^2. \qquad (67)
\end{aligned}$$

If $x(\tau - 1/\eta) \in B_{\xi'}(0)$, then (51) has an absolute value less than or equal to

$$KK_{vt}e^{4K/\eta}\left(\frac{2}{\eta}\right)^2\left\|x\left(\tau-\frac{1}{\eta}\right)\right\|^2. \tag{68}$$

(**V**) Having evaluated upper bounds for (50) and (51), we are set to evaluate (48). If $x(\tau - 1/\eta) \in B_{\xi'}(0)$, one obtains by combining (48), (62) and (68) that $\Delta V(\tau + 1/\eta, \tau - 1/\eta)$ is less than or equal to

$$\begin{aligned}
&\int_{\tau-1/\eta}^{\tau+1/\eta} \frac{\partial V}{\partial t}\left(x\left(\tau-\frac{1}{\eta}\right),t\right) + \frac{\partial V}{\partial x}\left(x\left(\tau-\frac{1}{\eta}\right),t\right) f\left(x\left(\tau-\frac{1}{\eta}\right),t,\eta^2 t\right)\,dt \\
&\qquad + L_1(\eta)\left(\frac{2}{\eta}\right)^2\left\|x\left(\tau-\frac{1}{\eta}\right)\right\|^2. \qquad (69)
\end{aligned}$$

Here, $L_1(\eta) = 2K^2K_v e^{4K/\eta} + KK_{vt}e^{4K/\eta}$ which decreases when η increases.

(**VI**) In order to invoke (8), (69) is rewritten such that $\Delta V(\tau + 1/\eta, \tau - 1/\eta)$ is less than or equal to

$$\int_{\tau-1/\eta}^{\tau+1/\eta} \frac{\partial V}{\partial t}\left(x\left(\tau - \frac{1}{\eta}\right), t\right) + \frac{\partial V}{\partial x}\left(x\left(\tau - \frac{1}{\eta}\right), t\right)\bar{f}\left(x\left(\tau - \frac{1}{\eta}\right), t\right) dt$$
$$+ \int_{\tau-1/\eta}^{\tau+1/\eta} \frac{\partial V}{\partial x}\left(x\left(\tau - \frac{1}{\eta}\right), t\right)\left(f\left(x\left(\tau - \frac{1}{\eta}\right), t, \eta^2 t\right) - \bar{f}\left(x\left(\tau - \frac{1}{\eta}\right), t\right)\right) dt$$
$$+ L_1(\eta)\left(\frac{2}{\eta}\right)^2 \left\| x\left(\tau - \frac{1}{\eta}\right)\right\|^2. \tag{70}$$

Here

$$\int_{\tau-1/\eta}^{\tau+1/\eta} \frac{\partial V}{\partial x}\left(x\left(\tau - \frac{1}{\eta}\right), t\right)\left(f\left(x\left(\tau - \frac{1}{\eta}\right), t, \eta^2 t\right) - \bar{f}\left(x\left(\tau - \frac{1}{\eta}\right), t\right)\right) dt$$
$$= \int_{\tau-1/\eta}^{\tau+1/\eta} \left(\frac{\partial V}{\partial x}\left(x\left(\tau - \frac{1}{\eta}\right), t\right) - \frac{\partial V}{\partial x}\left(x\left(\tau - \frac{1}{\eta}\right), \tau - \frac{1}{\eta}\right)\right)$$
$$\times \left(f\left(x\left(\tau - \frac{1}{\eta}\right), t, \eta^2 t\right) - \bar{f}\left(x\left(\tau - \frac{1}{\eta}\right), t\right)\right) dt$$
$$+ \frac{\partial V}{\partial x}\left(x\left(\tau - \frac{1}{\eta}\right), \tau - \frac{1}{\eta}\right) \int_{\tau-1/\eta}^{\tau+1/\eta} \left(f\left(x\left(\tau - \frac{1}{\eta}\right), t, \eta^2 t\right)\right.$$
$$\left. - \bar{f}\left(x\left(\tau - \frac{1}{\eta}\right), t\right)\right) dt. \tag{71}$$

By (63), it is clear that for all $t \in [\tau - 1/\eta, \tau + 1/\eta]$,

$$\left\| \frac{\partial V}{\partial x}\left(x\left(\tau - \frac{1}{\eta}\right), t\right) - \frac{\partial V}{\partial x}\left(x\left(\tau - \frac{1}{\eta}\right), \tau - \frac{1}{\eta}\right)\right\| \le K_{vxt}\frac{2}{\eta}\left\| x\left(\tau - \frac{1}{\eta}\right)\right\|. \tag{72}$$

By the Lipschitz property of f and $\bar{f}$,

$$\left\| f\left(x\left(\tau - \frac{1}{\eta}\right), t, \eta^2 t\right) - \bar{f}\left(x\left(\tau - \frac{1}{\eta}\right), t\right)\right\| \le (K + \bar{K})\left\| x\left(\tau - \frac{1}{\eta}\right)\right\|. \tag{73}$$

By (72), (73) and (5), one obtains that for all τ and for all $\eta \ge \eta_1$, (71) is less than or equal to

$$\left(\frac{2}{\eta}\right)^2 K_{vt}(K + \bar{K})\left\| x\left(\tau - \frac{1}{\eta}\right)\right\|^2 + K_v N(\eta)\left\| x\left(\tau - \frac{1}{\eta}\right)\right\|^2. \tag{74}$$

By combining (70), (8), (71) and (74) one obtains that for all $\eta \ge \eta_1$ and for all

$x(\tau - 1/\eta) \in B_{\xi'}(0)$,

$$\Delta V\left(\tau + \frac{1}{\eta}, \tau - \frac{1}{\eta}\right)$$
$$\leq \left(-\frac{2w}{\eta} + L_1(\eta)\left(\frac{2}{\eta}\right)^2 + K_v N(\eta) + \left(\frac{2}{\eta}\right)^2 K_{vt}(K + \bar{K})\right)\left\|x\left(\tau - \frac{1}{\eta}\right)\right\|^2 \tag{75}$$
$$= \frac{2}{\eta}\left(-w + L_1(\eta)\frac{2}{\eta} + K_v\frac{\eta}{2}N(\eta) + \frac{2}{\eta}K_{vt}(K + \bar{K})\right)\left\|x\left(\tau - \frac{1}{\eta}\right)\right\|^2. \tag{76}$$

(**VII**) At present, we are set to prove that $\Delta V(\tau + 1/\eta, \tau - 1/\eta)$ is negative. Since $\lim_{\eta\to\infty} \eta N(\eta) = 0$,

$$\lim_{\eta\to\infty} L_1(\eta)\frac{2}{\eta} + K_v\frac{\eta}{2}N(\eta) + \frac{2}{\eta}K_{vt}(K + \bar{K}) = 0. \tag{77}$$

One obtains the existence of an $\eta_2 > 0$ such that for all $\eta \geq \max\{\eta_1, \eta_2\}$,

$$\Delta V\left(\tau + \frac{1}{\eta}, \tau - \frac{1}{\eta}\right) \leq -\frac{w}{\eta}\left\|x\left(\tau - \frac{1}{\eta}\right)\right\|^2. \tag{78}$$

Here, τ was taken arbitrarily. Then $\Delta V(\tau + 1/\eta, \tau - 1/\eta) \leq -v\|x(\tau - 1/\eta)\|^2$ with $v = w/\eta$ when $x(\tau - 1/\eta) \in B_{\xi'}(0)$.

(**VIII**) By applying Theorem 1 with $U' = B_{\xi'}(0)$ and $t_k^* = \tau + (2k-1)/\eta$ for each $k \in \mathbb{Z}$, one obtains that the equilibrium point of (3) is locally exponentially stable for all $\alpha > \alpha_1 = \{\eta_1^2, \eta_2^2\}$.

Appendix B. Proof of Theorem 3

(**I**) The proof relies upon Theorem 2. When the partially averaged system $\dot{x}(t) = \bar{f}(x, t)$ is exponentially stable and (6) is satisfied, then the classical converse theorems of Liapunov [9, pp. 148–152] guarantee the existence of a Liapunov function $V(x, t)$ satisfying the inequalities

$$c_1\|x\|^2 \leq V(x, t) \leq c_2\|x\|^2, \tag{79}$$
$$\frac{\partial V}{\partial t}(x, t) + \frac{\partial V}{\partial x}(x, t)\bar{f}(x, t) \leq -c_3\|x\|^2, \tag{80}$$
$$\left\|\frac{\partial V}{\partial x}(x, t)\right\| \leq c_4\|x\| \tag{81}$$

for all $x \in B_{r_0}(0)$, for all $t \in \mathbb{R}$ and for some strictly positive contants c_1, c_2, c_3 and c_4.

Since (79) and (80) imply (7) and (8) with $v_{\min} = c_1$, $v_{\max} = c_2$, $w = c_3$, $U = B_{r_0}(0)$ and since Condition 2 of Theorem 3 implies Condition 1 of Theorem 2, only the smoothness conditions on $V(x, t)$ (Condition 2 of Theorem 2) have to be verified.

(**II**) In order to prove the Lipschitz property of $(\partial V/\partial x)(x,t)$ as required by Condition 2 of Theorem 2, we rely on Lemma 2.2, p. 71, of [9]. When $(\partial V/\partial x)(x,t)$ and its Jacobian matrix exist and are continuous on $B_{r_0}(0)$ and there exists a constant L_1 such that for all $x \in B_{r_0}(0)$ and for all $t \in \mathbb{R}$, $\|J_1(x,t)\| \le L_1$ where $J_1(x,t) = (J_{1ij}(x,t))$ with $J_{1ij}(x,t) = (\partial^2 V/\partial x_i\, \partial x_j)(x,t)$ then $(\partial V/\partial x)(x,t)$ is Lipschitz on $B_{r_0}(0)$ with Lipschitz constant L_1.

• First, we prove the existence and the continuity of $(\partial V/\partial x)(x,t)$ and its Jacobian. Since $\bar{f}(x,t)$ is continuous in t and three times continuously differentiable with respect to the state x, by Theorem 3.3, Exercise 3.2, pp. 21–22, of [7], $\Phi(\tau,t,x)$ $(\forall t,\ \forall \tau > t,\ \forall x \in B_{r_0}(0))$ is three times continuously differentiable with respect to x such that $\Phi_m(\tau,t,x)$, $(\partial \Phi_m/\partial x_i)(\tau,t,x)$ and $(\partial^2 \Phi_m/\partial x_i\, \partial x_j)(\tau,t,x)$ exist and are continuous in x and t for all $i,j,m \in \{1,\dots,n\}$. The function $V(x,t)$ in the converse theorem of Liapunov [9] is given as (with an appropriate T_V)

$$V(x,t) = \int_t^{t+T_V} \Phi(\tau,t,x)^T \Phi(\tau,t,x)\, dt = \sum_{m=1}^{n} \int_t^{t+T_V} \Phi_m^2(\tau,t,x)\, d\tau. \tag{82}$$

Taking the partial derivative of $V(x,t)$ with respect to x_i for all $i \in \{1,\dots,n\}$, one obtains that $\forall x \in B_{r_0}(0)$, $\forall t \in \mathbb{R}$ and $\forall i \in \{1,\dots,n\}$,

$$\frac{\partial V}{\partial x_i}(x,t) = \sum_{m=1}^{n} \int_t^{t+T_V} 2 \frac{\partial \Phi_m}{\partial x_i}(\tau,t,x) \Phi_m(\tau,t,x)\, d\tau. \tag{83}$$

Taking the partial derivative of (83) with respect to x_j, one obtains that $\forall x \in B_{r_0}(0)$, $\forall t \in \mathbb{R}$ and $\forall i,j \in \{1,\dots,n\}$,

$$\begin{aligned}\frac{\partial^2 V}{\partial x_i\, \partial x_j}(x,t) = 2\sum_{m=1}^{n} \int_t^{t+T_V} & \frac{\partial^2 \Phi_m}{\partial x_i\, \partial x_j}(\tau,t,x)\Phi_m(\tau,t,x) \\ & + \frac{\partial \Phi_m}{\partial x_i}(\tau,t,x)\frac{\partial \Phi_m}{\partial x_j}(\tau,t,x)\, d\tau. \end{aligned} \tag{84}$$

The existence and continuity of $\Phi_m(\tau,t,x)$, $(\partial \Phi_m/\partial x_i)(\tau,t,x)$ and $(\partial^2 \Phi_m/\partial x_i\, \partial x_j) \cdot (\tau,t,x)$ for all $i,j,m \in \{1,\dots,n\}$ imply by (82), (83) and (84) that $(\partial V/\partial x_i)(x,t)$ and $(\partial^2 V/\partial x_i\, \partial x_j)(x,t)$ exist and are continuous in x and t for all $x \in B_{r_0}(0)$ and all $t \in \mathbb{R}$.

• Second, we prove boundedness of the Jacobian of $(\partial V/\partial x)(x,t)$. Appendix D examines the flow of the partially averaged system. Relying on (84), Appendix E examines the Liapunov function $V(x,t)$. By (127) in Appendix E,

$$\begin{aligned}\sum_{i,j=1}^{n} |J_{1ij}(x,t)| &= \sum_{i,j=1}^{n} \left| \frac{\partial^2 V}{\partial x_i\, \partial x_j}(x,t) \right| \\ &\le 2T_V(M_5(T_V)k\|x\| + M_4^2(T_V)) =: M_9(\|x\|). \end{aligned} \tag{85}$$

This implies that the Jacobian matrix $J_1(x,t) = (J_{1ij}(x,t))$ of $(\partial V/\partial x)(x,t)$ is bounded on $B_{r_0}(0)$, i.e., for all $x \in B_{r_0}(0)$ and all $t \in \mathbb{R}$, $\|J_1(x,t)\| \le M_9(\|x\|) \le M_9(r_0)$.

• By Lemma 2.2, p. 71, of [9], $(\partial V/\partial x)(x,t)$ is Lipschitz on $B_{r_0}(0)$ with Lipschitz constant $L_1 = M_9(r_0)$. By this Lipschitz condition and by (81), $(\partial V/\partial x)(0,t) = 0$ for all $t \in \mathbb{R}$, such that Condition 2 of Theorem 2 is satisfied with $U = B_{r_0}(0)$.

(**III**) In order to prove the Lipschitz property of $(\partial^2 V/\partial x\,\partial t)(x,t)$ as required by Condition 2 of Theorem 2, we rely on Lemma 2.2, p. 71, of [9]. When $(\partial^2 V/\partial x\,\partial t)(x,t)$ and its Jacobian matrix exist and are continuous on $B_{r_0}(0)$ and there exists a constant L_2 such that for all $x \in B_{r_0}(0)$ and for all $t \in \mathbb{R}$, $\|J_2(x,t)\| \le L_2$ where $J_2(x,t) = (J_{2ij}(x,t))$ with $J_{2ij}(x,t) = (\partial^3 V/\partial x_i\,\partial x_j\,\partial t)(x,t)$ then $(\partial^2 V/\partial x\,\partial t)(x,t)$ is Lipschitz on $B_{r_0}(0)$ with Lipschitz constant L_2.

• First, we prove the existence and continuity of $(\partial^2 V/\partial x\,\partial t)(x,t)$ and its Jacobian. Since $\bar{f}(x,t)$ is continuous in t and three times continuously differentiable with respect to the state x, by applying Theorem 3.3, Exercise 3.2, pp. 21–22, of [7], $\Phi(\tau,t,x)$ $(\forall t,\ \forall\tau > t,\ \forall x \in B_{r_0}(0))$ is three times continuously differentiable with respect to x such that $\Phi_m(\tau,t,x)$, $(\partial\Phi_m/\partial x_i)(\tau,t,x)$, $(\partial^2\Phi_m/\partial x_i\,\partial x_j)(\tau,t,x)$ and $(\partial^3\Phi_m/\partial x_i\,\partial x_j\,\partial x_l)(\tau,t,x)$ exist and are continuous in x and t for all $i,j,m,l \in \{1,\dots,n\}$.

At present, we are set to prove the existence and continuity of $(\partial\Phi_m/\partial t)(\tau,t,x)$, $(\partial^2\Phi_m/\partial x_i\,\partial t)(\tau,t,x)$, $(\partial^3\Phi_m/\partial x_i\,\partial x_j\,\partial t)(\tau,t,x)$. The uniqueness of the solutions implies that $\forall m \in \{1,\dots,n\}$, $\forall t$, $\forall\tau > t$ and $\forall x \in B_{r_0}(0)$,

$$\Phi_m(\tau,t,x) = \Phi_m(\tau,t+h,\Phi(t+h,t,x)) \tag{86}$$

for h sufficiently small. By (86), it is obvious that for all $i,j,m \in \{1,\dots,n\}$,

$$\begin{aligned} &h^{-1}(\Phi_m(\tau,t+h,x) - \Phi_m(\tau,t,x)) \\ &\qquad = h^{-1}(\Phi_m(\tau,t+h,x) - \Phi_m(\tau,t+h,\Phi(t+h,t,x))), \end{aligned} \tag{87}$$

$$\begin{aligned} &h^{-1}\left(\frac{\partial\Phi_m}{\partial x_i}(\tau,t+h,x) - \frac{\partial\Phi_m}{\partial x_i}(\tau,t,x)\right) \\ &\qquad = h^{-1}\left(\frac{\partial\Phi_m}{\partial x_i}(\tau,t+h,x) - \frac{\partial\Phi_m}{\partial x_i}(\tau,t+h,\Phi(t+h,t,x))\right) \end{aligned} \tag{88}$$

and that

$$\begin{aligned} &h^{-1}\left(\frac{\partial^2\Phi_m}{\partial x_i\,\partial x_j}(\tau,t+h,x) - \frac{\partial^2\Phi_m}{\partial x_i\,\partial x_j}(\tau,t,x)\right) \\ &\qquad = h^{-1}\left(\frac{\partial^2\Phi_m}{\partial x_i\,\partial x_j}(\tau,t+h,x) - \frac{\partial^2\Phi_m}{\partial x_i\,\partial x_j}(\tau,t+h,\Phi(t+h,t,x))\right). \end{aligned} \tag{89}$$

The limit of the right-hand side of (87) as $h \to 0$ equals $-(\partial\Phi_m/\partial x)(\tau,t,x)\bar{f}(t,x)$ which is continuous in x and t. Therefore, the limit of the left-hand side of (87) as $h \to 0$ exists and is also continuous in x and t. This limit of the left-hand side of (87) as $h \to 0$ equals $(\partial\Phi_m/\partial t)(\tau,t,x)$.

The limit of the right-hand side of (88) as $h \to 0$ equals $-(\partial/\partial x)\cdot((\partial\Phi_m/\partial x_i)(\tau,t,x))\bar{f}(t,x)$ which is continuous in x and t. Therefore, the limit of the left-hand side of (88) as $h \to 0$ exists and is also continuous in x and t. This limit of the left-hand side of (88) as $h \to 0$ equals $(\partial^2\Phi_m/\partial x_i\,\partial t)(\tau,t,x)$.

The limit of the right-hand side of (89) as $h \to 0$ equals $-(\partial/\partial x)\cdot((\partial^2\Phi_m/\partial x_i\,\partial x_j)(\tau,t,x))\bar{f}(t,x)$ which is continuous in x and t. Therefore, the limit of the left-hand side of (89) as $h \to 0$ exists and is also continuous in x and t. This limit of the left-hand side of (89) as $h \to 0$ equals $(\partial^3\Phi_m/\partial x_i\,\partial x_j\,\partial t)(\tau,t,x)$.

Therefore, not only $\Phi_m(\tau,t,x)$, $(\partial\Phi_m/\partial x_i)(\tau,t,x)$, $(\partial^2\Phi_m/\partial x_i\,\partial x_j)(\tau,t,x)$ and $(\partial^3\Phi_m/\partial x_i\,\partial x_j\,\partial x_l)(\tau,t,x)$, but also $(\partial\Phi_m/\partial t)(\tau,t,x)$, $(\partial^2\Phi_m/\partial x_i\,\partial t)(\tau,t,x)$ and $(\partial^3\Phi_m/\partial x_i\,\partial x_j\,\partial t)(\tau,t,x)$ exist and are continuous in x and t.

At present, we are set to prove the existence and continuity of $(\partial^2 V/\partial x\,\partial t)(x,t)$ and its Jacobian. Taking the partial derivative of (83) with respect to t, one obtains that $\forall x \in B_{r_0}(0)$, $\forall t \in \mathbb{R}$ and $\forall i \in \{1,\ldots,n\}$,

$$\begin{aligned}\frac{\partial^2 V}{\partial x_i\,\partial t}(x,t) &= -2x_i + 2\sum_{m=1}^{n}\frac{\partial\Phi_m}{\partial x_i}(t+T_V,t,x)\Phi_m(t+T_V,t,x)\\ &\quad+\sum_{m=1}^{n}\int_t^{t+T_V} 2\frac{\partial^2\Phi_m}{\partial x_i\,\partial t}(\tau,t,x)\Phi_m(\tau,t,x)\\ &\quad+2\frac{\partial\Phi_m}{\partial x_i}(\tau,t,x)\frac{\partial\Phi_m}{\partial t}(\tau,t,x)\,d\tau. \end{aligned}\tag{90}$$

By taking the partial derivative of (84) with respect to t, one obtains that

$$\begin{aligned}\frac{\partial^3 V}{\partial x_i\,\partial x_j\,\partial t}(x,t) &= -2\delta_{ij} + 2\sum_{m=1}^{n}\Bigg(\frac{\partial^2\Phi_m}{\partial x_i\,\partial x_j}(t+T_V,t,x)\Phi_m(t+T_V,t,x)\\ &\qquad+\frac{\partial\Phi_m}{\partial x_i}(t+T_V,t,x)\frac{\partial\Phi_m}{\partial x_j}(t+T_V,t,x)\Bigg)\\ &\quad+2\sum_{m=1}^{n}\int_t^{t+T_V}\frac{\partial^3\Phi_m}{\partial x_i\,\partial x_j\,\partial t}(\tau,t,x)\Phi_m(\tau,t,x)\\ &\quad+\frac{\partial^2\Phi_m}{\partial x_i\,\partial x_j}(\tau,t,x)\frac{\partial\Phi_m}{\partial t}(\tau,t,x)\,d\tau\\ &\quad+2\sum_{m=1}^{n}\int_t^{t+T_V}\frac{\partial^2\Phi_m}{\partial x_i\,\partial t}(\tau,t,x)\frac{\partial\Phi_m}{\partial x_j}(\tau,t,x)\\ &\quad+\frac{\partial\Phi_m}{\partial x_i}(\tau,t,x)\frac{\partial^2\Phi_m}{\partial x_j\,\partial t}(\tau,t,x)\,d\tau. \end{aligned}\tag{91}$$

The existence and continuity of $\Phi_m(\tau,t,x)$, $(\partial\Phi_m/\partial x_i)(\tau,t,x)$, $(\partial^2\Phi_m/\partial x_i\,\partial x_j)\cdot(\tau,t,x)$, $(\partial\Phi_m/\partial t)(\tau,t,x)$, $(\partial^2\Phi_m/\partial x_i\,\partial t)(\tau,t,x)$ and $(\partial^3\Phi_m/\partial x_i\,\partial x_j\,\partial t)(\tau,t,x)$ implies by (90) and (91) that $(\partial^2 V/\partial x_i\,\partial t)(x,t)$ and $(\partial^3 V/\partial x_i\,\partial x_j\,\partial t)(x,t)$ exist and are continuous in x and t for all $x \in B_{r_0}(0)$ and for all $t \in \mathbb{R}$.

- Second, we prove boundedness of the Jacobian of $(\partial^2 V/\partial x\,\partial t)(x,t)$. Appendix

D studies the flow of the partially averaged system. Relying on (91), Appendix E studies the Liapunov function $V(x,t)$ and by (132) in Appendix E,

$$\sum_{i,j=1}^{n} |J_{2ij}(x,t)| = \sum_{i,j=1}^{n} \left| \frac{\partial^3 V}{\partial x_i\, \partial x_j\, \partial t}(x,t) \right| \le M_{10}(\|x\|)$$
$$:= 2n + 2M_5(T_V)k\|x\| + 2M_4^2(T_V) + 2T_V(M_8(T_V, \|x\|)k\|x\|$$
$$+ M_5(T_V)M_6(T_V, \|x\|) + 2M_4(T_V)M_7(T_V, \|x\|)). \tag{92}$$

This implies that the Jacobian matrix $J_2(x,t) = (J_{2ij}(x,t))$ of $(\partial^2 V/\partial x\, \partial t)(x,t)$ is bounded on $B_{r_0}(0)$, i.e., for all $x \in B_{r_0}(0)$ and for all $t \in \mathbb{R}$, $\|J_2(x,t)\| \le M_{10}(\|x\|) \le M_{10}(r_0)$.

- By Lemma 2.2, p. 71, of [9], $(\partial^2 V/\partial x\, \partial t)(x,t)$ is Lipschitz on $B_{r_0}(0)$ with Lipschitz constant $L_2 = M_{10}(r_0)$. By this Lipschitz condition and by (90), $(\partial^2 V/\partial x\, \partial t) \cdot (0,t) = 0$ for all $t \in \mathbb{R}$, such that Condition 2 of Theorem 2 is satisfied with $U = B_{r_0}(0)$.

(**IV**) Since $\bar{f}(x,t)$ and its Jacobian $(\partial\bar{f}/\partial x)(x,t)$ exist and since they are continuous [9, Lemma 2.2, p. 71], the boundedness of $\|(\partial\bar{f}/\partial x)(x,t)\|$ implies that $\bar{f}(x,t)$ is Lipschitz in $B_{r_0}(0)$.

(**V**) By (**I**)–(**IV**), the conditions of Theorem 2 are satisfied with $U = B_{r_0}(0)$ such that Theorem 2 implies the existence of an $\alpha_1 > 0$ such that for all $\alpha > \alpha_1$ the equilibrium point $x = 0$ of the system (3) is locally exponentially stable.

Appendix C. Proof of Theorem 4

The proof relies upon Theorem 3. Notice first that by Conditions 1 and 4 of Theorem 4, Conditions 1 and 3 of Theorem 3 are satisfied. Relying on Conditions 2 and 3 of Theorem 4, we will prove that Condition 2 of Theorem 3 is satisfied. Since $\forall x \in B_{r_0}(0)$, $\forall \tau_1, \tau_2 \in \mathbb{R}$ and $\forall \eta > 0$,

$$\frac{1}{2\eta}\int_{\eta^2\tau_1-\eta}^{\eta^2\tau_1+\eta} (f(x,\tau_2,t) - \bar{f}(x,\tau_2))\, dt = \frac{\eta}{2}\int_{\tau_1-1/\eta}^{\tau_1+1/\eta} (f(x,\tau_2,\eta^2 t) - \bar{f}(x,\tau_2))\, dt, \tag{93}$$

one obtains by (26) that $\forall \tau_1, \tau_2 \in \mathbb{R}$, $\forall \eta > 0$ and $\forall x \in B_{r_0}(0) \subset W$,

$$\left\| \int_{\tau_1-1/\eta}^{\tau_1+1/\eta} (f(x,\tau_2,\eta^2 t) - \bar{f}(x,\tau_2))\, dt \right\| \le \frac{M(2\eta)}{\eta^2}\|x\|. \tag{94}$$

Take $\eta_1 = 2/l$ such that for all $\tau \in \mathbb{R}$ and for all $\eta \ge \eta_1$ the open interval $(\tau - 1/\eta, \tau + 1/\eta)$ contains maximum n_{d} discontinuities of $f(x,\tau,t) - \bar{f}(x,\tau)$ in τ for every fixed x and t. We denote the exact number of discontinuities in $(\tau - 1/\eta, \tau + 1/\eta)$ as $n_{\mathrm{d}}(\tau,\eta)$ and the discontinuities as $\tau_1(\tau,\eta), \ldots, \tau_{n_{\mathrm{d}}(\tau,\eta)}(\tau,\eta)$. We set $\tau_0(\tau,\eta) = \tau - 1/\eta$ and $\tau_{n_{\mathrm{d}}(\tau,\eta)+1}(\tau,\eta) = \tau + 1/\eta$. Therefore, $\forall \tau \in \mathbb{R}$, $\forall \eta \ge \eta_1$

and $\forall x \in B_{r_0}(0)$,

$$\left\| \int_{\tau-1/\eta}^{\tau+1/\eta} (f(x,t,\eta^2 t) - \bar{f}(x,t))\, dt \right\| \leq \sum_{i=0}^{n_{\mathrm{d}}(\tau,\eta)} \left\| \int_{\tau_i(\tau,\eta)}^{\tau_{i+1}(\tau,\eta)} (f(x,t,\eta^2 t) - \bar{f}(x,t)\, dt \right\|. \quad (95)$$

It is clear that (95) is less than or equal to

$$\sum_{i=0}^{n_{\mathrm{d}}(\tau,\eta)} \left\| \int_{\tau_i(\tau,\eta)}^{\tau_{i+1}(\tau,\eta)} (f(x,\bar{\tau}_i(\tau,\eta),\eta^2 t) - \bar{f}(x,\bar{\tau}_i(\tau,\eta)))\, dt \right\| + \sum_{i=0}^{n_{\mathrm{d}}(\tau,\eta)} \left\| \int_{\tau_i(\tau,\eta)}^{\tau_{i+1}(\tau,\eta)} ((f(x,t,\eta^2 t) - \bar{f}(x,t)) - (f(x,\bar{\tau}_i(\tau,\eta),\eta^2 t) - \bar{f}(x,\bar{\tau}_i(\tau,\eta))))\, dt \right\| \quad (96)$$

where $\bar{\tau}_i(\tau,\eta) = (\tau_i(\tau,\eta) + \tau_{i+1}(\tau,\eta))/2$ for all $i \in \{0,\ldots,n_{\mathrm{d}}(\tau,\eta)\}$. By applying (27), it is obvious that for all $t \in (\tau_i(\tau,\eta),\tau_{i+1}(\tau,\eta))$ and for all $x \in B_{r_0}(0)$,

$$\|(f(x,t,\eta^2 t) - \bar{f}(x,t)) - (f(x,\bar{\tau}_i(\tau,\eta),\eta^2 t) - \bar{f}(x,\bar{\tau}_i(\tau,\eta)))\| \leq K_{r_0}\left(\frac{\tau_{i+1}(\tau,\eta) - \tau_i(\tau,\eta)}{2}\right)\|x\|. \quad (97)$$

Here, $\tau_1 = \bar{\tau}_i(\tau,\eta)$, $\tau_2 = t$ and $t_1 = \eta^2 t$ when $t \geq \bar{\tau}_i(\tau,\eta)$. When $t < \bar{\tau}_i(\tau,\eta)$, then $\tau_1 = t$, $\tau_2 = \bar{\tau}_i(\tau,\eta)$ and $t_1 = \eta^2 t$. By (93), it is clear that

$$\sum_{i=0}^{n_{\mathrm{d}}(\tau,\eta)} \left\| \int_{\tau_i(\tau,\eta)}^{\tau_{i+1}(\tau,\eta)} (f(x,\bar{\tau}_i(\tau,\eta),\eta^2 t) - \bar{f}(x,\bar{\tau}_i(\tau,\eta)))\, dt \right\| \leq \sum_{i=0}^{n_{\mathrm{d}}(\tau,\eta)} \frac{1}{\eta^2} \left\| \int_{\eta^2\tau_i(\tau,\eta)}^{\eta^2\tau_{i+1}(\tau,\eta)} (f(x,\bar{\tau}_i(\tau,\eta),t) - \bar{f}(x,\bar{\tau}_i(\tau,\eta)))\, dt \right\|. \quad (98)$$

By applying (96), (98), (26) and (97), one obtains that for all $\tau \in \mathbb{R}$, all $\eta \geq \eta_1$ and all $x \in B_{r_0}(0)$, (95) is less than or equal to

$$\sum_{i=0}^{n_{\mathrm{d}}(\tau,\eta)} \left(\frac{M(\eta^2\tau_{i+1}(\tau,\eta) - \eta^2\tau_i(\tau,\eta))}{\eta^2}\|x\| + (\tau_{i+1}(\tau,\eta) - \tau_i(\tau,\eta)) K_{r_0}\left(\frac{\tau_{i+1}(\tau,\eta) - \tau_i(\tau,\eta)}{2}\right)\|x\| \right). \quad (99)$$

Since $\tau_{i+1}(\tau,\eta) - \tau_i(\tau,\eta) \leq 2/\eta$, $n_{\mathrm{d}}(\tau,\eta) \leq n_{\mathrm{d}}$ and since $M(\cdot)$ and $K_{r_0}(\cdot)$ are non-decreasing functions, one obtains that $\forall \tau \in \mathbb{R}$, $\forall \eta \geq \eta_1$ and $\forall x \in B_{r_0}(0)$,

$$\left\| \int_{\tau-1/\eta}^{\tau+1/\eta} (f(x,t,\eta^2 t) - \bar{f}(x,t))\, dt \right\| \leq (n_{\mathrm{d}} + 1)\frac{M(2\eta)}{\eta^2}\|x\| + (n_{\mathrm{d}} + 1)\frac{2}{\eta} K_{r_0}\left(\frac{1}{\eta}\right)\|x\|. \quad (100)$$

By definining $N(\eta) := (n_{\mathrm{d}} + 1)(M(2\eta)/\eta^2 + (2/\eta)K_{r_0}(1/\eta))$, Condition 2 of Theorem 3 is satisfied since $\lim_{\eta\to 0} \eta N(\eta) = 0$.

By applying Theorem 3, there exists an $\alpha_1 > 0$ such that for all $\alpha > \alpha_1$ the equilibrium point $x = 0$ of the system (3) is exponentially stable.

Appendix D. The Flow of the Partially Averaged System

The solution $\Phi(\tau, t, x)$ of the partially averaged system $\dot{x} = \bar{f}(x, t)$ satisfies for all $x \in B_{r_0}(0)$ and all $\tau, t \in \mathbb{R}$ with $\tau \geq t$,

$$\Phi(\tau, t, x) = x + \int_t^\tau \bar{f}(\Phi(s, t, x), s)\, ds. \tag{101}$$

With $\Phi = (\Phi_1, \dots, \Phi_n)^T$, it is obvious that for all $i \in \{1, \dots, n\}$,

$$\Phi_i(\tau, t, x) = x_i + \int_t^\tau \bar{f}_i(\Phi_1(s, t, x), \dots, \Phi_n(s, t, x), s)\, ds. \tag{102}$$

(**I**) Taking the partial derivative with respect to x_j, one obtains that

$$\frac{\partial \Phi_i}{\partial x_j}(\tau, t, x) = \delta_{ij} + \int_t^\tau \sum_{m=1}^n \frac{\partial \bar{f}_i}{\partial \Phi_m}(\Phi(s, t, x), s)\frac{\partial \Phi_m}{\partial x_j}(s, t, x)\, ds. \tag{103}$$

Taking the absolute value of (103), invoking the triangle inequality and taking the sum of (103) over all $i, j \in \{1, \dots, n\}$, one obtains by (6), with $\Phi(s, t, x) \in B_{kr_0}(0)$, that

$$\sum_{i,j=1}^n \left|\frac{\partial \Phi_i}{\partial x_j}(\tau, t, x)\right| \leq \sum_{i,j=1}^n \delta_{ij} + M_1 \sum_{i,j=1}^n \int_t^\tau \sum_{m=1}^n \left|\frac{\partial \Phi_m}{\partial x_j}(s, t, x)\right| ds \tag{104}$$

and

$$\sum_{i,j=1}^n \left|\frac{\partial \Phi_i}{\partial x_j}(\tau, t, x)\right| \leq n + nM_1 \int_t^\tau \sum_{m,j=1}^n \left|\frac{\partial \Phi_m}{\partial x_j}(s, t, x)\right| ds. \tag{105}$$

Application of the Bellman–Gronwall lemma gives for all $\tau, t \in \mathbb{R}$ with $\tau \geq t$ that

$$\sum_{i,j=1}^n \left|\frac{\partial \Phi_i}{\partial x_j}(\tau, t, x)\right| \leq n e^{nM_1(\tau - t)} \leq n e^{nM_1 T'} =: M_4(T') \tag{106}$$

when $T' \geq \tau - t$.

(**II**) The partial derivative of the right-hand side of (103) with respect to x_l equals

$$\sum_{m=1}^n \int_t^\tau \left(\sum_{p=1}^n \frac{\partial^2 \bar{f}_i}{\partial \Phi_m\, \partial \Phi_p}(\Phi(s, t, x), s)\frac{\partial \Phi_p}{\partial x_l}(s, t, x)\right)\frac{\partial \Phi_m}{\partial x_j}(s, t, x) + \frac{\partial \bar{f}_i}{\partial \Phi_m}(\Phi(s, t, x), s)\frac{\partial^2 \Phi_m}{\partial x_j\, \partial x_l}(s, t, x)\, ds. \tag{107}$$

Taking the absolute value, invoking the triangle inequality and by (6) and (106), one obtains with $\Phi(s, t, x) \in B_{kr_0}(0)$ that for all $i, j, l \in \{1, \dots, n\}$ and all $\tau, t \in \mathbb{R}$

with $\tau \geq t$ that

$$\left|\frac{\partial^2 \Phi_i}{\partial x_j\, \partial x_l}(\tau, t, x)\right| \leq \sum_{m=1}^{n} \int_t^{\tau} M_2 M_4^2(T') + M_1 \left|\frac{\partial^2 \Phi_m}{\partial x_j\, \partial x_l}(s, t, x)\right| ds \tag{108}$$

when $T' \geq \tau - t$. Taking the sum of (108) over all $i, j, l \in \{1, \dots, n\}$ leads to

$$\sum_{i,j,l=1}^{n} \left|\frac{\partial^2 \Phi_i}{\partial x_j\, \partial x_l}(\tau, t, x)\right| \leq (\tau - t) n^4 M_2 M_4^2(T') + n M_1 \sum_{m,j,l=1}^{n} \int_t^{\tau} \left|\frac{\partial^2 \Phi_m}{\partial x_j\, \partial x_l}(s, t, x)\right| ds. \tag{109}$$

Application of the Bellman–Gronwall lemma gives for all $\tau, t \in \mathbb{R}$ with $\tau \geq t$ that

$$\sum_{i,j,l=1}^{n} \left|\frac{\partial^2 \Phi_i}{\partial x_j\, \partial x_l}(\tau, t, x)\right| \leq T' n^4 M_2 M_4^2(T') e^{n M_1 (\tau - t)} \leq T' n^4 M_2 M_4^2(T') e^{n M_1 T'} =: M_5(T') \tag{110}$$

when $T' \geq \tau - t$.

(**III**) Taking the partial derivative with respect to t of (102), one obtains that

$$\frac{\partial \Phi_i}{\partial t}(\tau, t, x) = -\bar{f}_i(x, t) + \int_t^{\tau} \sum_{m=1}^{n} \frac{\partial \bar{f}_i}{\partial \Phi_m}(\Phi(s, t, x), s) \frac{\partial \Phi_m}{\partial t}(s, t, x)\, ds. \tag{111}$$

Taking the absolute value of (111), invoking the triangle inequality and taking the sum of (111) over all $i \in \{1, \dots, n\}$, one obtains by (6) with $\Phi(s, t, x) \in B_{kr_0}(0)$ that

$$\sum_{i=1}^{n} \left|\frac{\partial \Phi_i}{\partial t}(\tau, t, x)\right| \leq F\|x\| + n M_1 \int_t^{\tau} \sum_{m=1}^{n} \left|\frac{\partial \Phi_m}{\partial t}(s, t, x)\right| ds. \tag{112}$$

The Lipschitz property of $\bar{f}(x, t)$ implies the existence of a $F > 0$ such that for all $x \in B_{r_0}(0)$ and all $t \in \mathbb{R}$, $\sum_{i=1}^{n} |f_i(x, t)| \leq F\|x\|$. Application of the Bellman–Gronwall lemma gives for all $\tau, t \in \mathbb{R}$ with $\tau \geq t$ that

$$\sum_{i=1}^{n} \left|\frac{\partial \Phi_i}{\partial t}(\tau, t, x)\right| \leq F\|x\| e^{n M_1 (\tau - t)} \leq F\|x\| e^{n M_1 T'} =: M_6(T', \|x\|) \tag{113}$$

when $T' \geq \tau - t$.

(**IV**) The partial derivative of the right-hand side of (103) with respect to t equals

$$-\sum_{m=1}^{n} \frac{\partial \bar{f}_i}{\partial x_m}(x, t) \delta_{mj} + \sum_{m=1}^{n} \int_t^{\tau} \left(\sum_{p=1}^{n} \frac{\partial^2 \bar{f}_i}{\partial \Phi_m\, \partial \Phi_p}(\Phi(s, t, x), s) \frac{\partial \Phi_p}{\partial t}(s, t, x)\right) \frac{\partial \Phi_m}{\partial x_j}(s, t, x) + \frac{\partial \bar{f}_i}{\partial \Phi_m}(\Phi(s, t, x), s) \frac{\partial^2 \Phi_m}{\partial x_j\, \partial t}(s, t, x)\, ds. \tag{114}$$

Taking the absolute value, invoking the triangle inequality and by (6), (106) and (113), one obtains with $\Phi(s, t, x) \in B_{kr_0}(0)$ that for all $i, j \in \{1, \dots, n\}$ and all

$\tau, t \in \mathbb{R}$ with $\tau \geq t$ that

$$\left|\frac{\partial^2 \Phi_i}{\partial x_j\, \partial t}(\tau, t, x)\right| \leq M_1 + \sum_{m=1}^{n} \int_t^\tau M_2 M_6(T', \|x\|) M_4(T') + M_1 \left|\frac{\partial^2 \Phi_m}{\partial x_j\, \partial t}(s, t, x)\right| ds \tag{115}$$

when $T' \geq \tau - t$. Taking the sum of (115) over all $i, j \in \{1, \ldots, n\}$ leads to

$$\sum_{i,j=1}^{n} \left|\frac{\partial^2 \Phi_i}{\partial x_j\, \partial t}(\tau, t, x)\right| \leq n^2 M_1 + T' n^3 M_2 M_6(T', \|x\|) M_4(T') + n M_1 \sum_{m,j=1}^{n} \int_t^\tau \left|\frac{\partial^2 \Phi_m}{\partial x_j\, \partial t}(s, t, x)\right| ds \tag{116}$$

when $T' \geq \tau - t$. Application of the Bellman–Gronwall lemma gives for all $\tau, t \in \mathbb{R}$ with $\tau \geq t$ that

$$\begin{aligned}\sum_{i,j=1}^{n} \left|\frac{\partial^2 \Phi_i}{\partial x_j\, \partial t}(\tau, t, x)\right| &\leq (n^2 M_1 + T' n^3 M_2 M_6(T', \|x\|) M_4(T')) e^{n M_1 (\tau - t)} \\ &\leq (n^2 M_1 + T' n^3 M_2 M_6(T', \|x\|) M_4(T')) e^{n M_1 T'} \\ &=: M_7(T', \|x\|)\end{aligned} \tag{117}$$

when $T' \geq \tau - t$.

(**V**) The partial derivative of (107) with respect to t, which is also equal to $(\partial^3 \Phi_i / \partial x_j\, \partial x_l\, \partial t)(\tau, t, x)$, equals

$$\begin{aligned}
&-\sum_{m=1}^{n}\left(\left(\sum_{p=1}^{n} \frac{\partial^2 \bar{f}_i}{\partial x_m\, \partial x_p}(x, t)\delta_{pl}\right)\delta_{mj}\right) \\
&\quad + \sum_{m=1}^{n} \int_t^\tau \left(\sum_{p=1}^{n}\left(\sum_{r=1}^{n} \frac{\partial^3 \bar{f}_i}{\partial \Phi_m\, \partial \Phi_p\, \partial \Phi_r}(\Phi(s, t, x), s) \frac{\partial \Phi_r}{\partial t}(s, t, x)\right) \frac{\partial \Phi_p}{\partial x_l}(s, t, x)\right) \\
&\quad \times \frac{\partial \Phi_m}{\partial x_j}(s, t, x)\, ds \\
&\quad + \sum_{m=1}^{n} \int_t^\tau \left(\sum_{p=1}^{n} \frac{\partial^2 \bar{f}_i}{\partial \Phi_m\, \partial \Phi_p}(\Phi(s, t, x), s) \frac{\partial^2 \Phi_p}{\partial x_l\, \partial t}(s, t, x)\right) \frac{\partial \Phi_m}{\partial x_j}(s, t, x)\, ds \\
&\quad + \sum_{m=1}^{n} \int_t^\tau \left(\sum_{p=1}^{n} \frac{\partial^2 \bar{f}_i}{\partial \Phi_m\, \partial \Phi_p}(\Phi(s, t, x), s) \frac{\partial \Phi_p}{\partial x_l}(s, t, x)\right) \frac{\partial^2 \Phi_m}{\partial x_j\, \partial t}(s, t, x)\, ds \\
&\quad + \sum_{m=1}^{n} \int_t^\tau \left(\sum_{r=1}^{n} \frac{\partial^2 \bar{f}_i}{\partial \Phi_m\, \partial \Phi_r}(\Phi(s, t, x), s) \frac{\partial \Phi_r}{\partial t}(s, t, x)\right) \frac{\partial^2 \Phi_m}{\partial x_j\, \partial x_l}(s, t, x)\, ds \\
&\quad + \sum_{m=1}^{n} \int_t^\tau \frac{\partial \bar{f}_i}{\partial \Phi_m}(\Phi(s, t, x), s) \frac{\partial^3 \Phi_m}{\partial x_j\, \partial x_l\, \partial t}(s, t, x)\, ds.
\end{aligned} \tag{118}$$

Taking the absolute value, invoking the triangle inequality and by (6), (106), (110), (113) and (117), one obtains with $\Phi(s,t,x) \in B_{kr_0}(0)$ that for all $i,j,l \in \{1,\ldots,n\}$ and all $\tau, t \in \mathbb{R}$ with $\tau \geq t$ that

$$\begin{aligned} \left|\frac{\partial^3 \Phi_i}{\partial x_j\,\partial x_l\,\partial t}(\tau,t,x)\right| \leq{} & \sum_{m=1}^{n}\int_t^{\tau} n^2 M_3 M_6(T',\|x\|)M_4^2(T')\,ds \\ & + \sum_{m=1}^{n}\int_t^{\tau} 2nM_2M_7(T',\|x\|)M_4(T') \\ & + nM_2M_6(T',\|x\|)M_5(T')\,ds \\ & + M_3 + \sum_{m=1}^{n}\int_t^{\tau} M_1\left|\frac{\partial^3\Phi_m}{\partial x_j\,\partial x_l\,\partial t}(s,t,x)\right| ds \end{aligned} \tag{119}$$

when $T' \geq \tau - t$. Taking the sum of (119) over all $i,j,l \in \{1,\ldots,n\}$ leads to

$$\sum_{i,j,l=1}^{n}\left|\frac{\partial^3 \Phi_i}{\partial x_j\,\partial x_l\,\partial t}(\tau,t,x)\right|$$

$$\leq n^6 T' M_3 M_6(T',\|x\|)M_4^2(T') \tag{120}$$

$$+ 2n^5 T' M_2 M_7(T',\|x\|)M_4(T') + n^5 T' M_2 M_6(T')M_5(T') \tag{121}$$

$$+ n^3 M_3 + nM_1 \sum_{m,j,l=1}^{n}\left|\frac{\partial^3\Phi_m}{\partial x_j\,\partial x_l\,\partial t}(s,t,x)\right| ds \tag{122}$$

when $T' \geq \tau - t$. Application of the Bellman–Gronwall lemma gives for all $\tau, t \in \mathbb{R}$ with $\tau \geq t$ that

$$\begin{aligned} \sum_{i,j,l=1}^{n}\left|\frac{\partial^3 \Phi_i}{\partial x_j\,\partial x_l\,\partial t}(\tau,t,x)\right| &\leq M_8'(T',\|x\|)e^{nM_1(\tau-t)} \\ &\leq M_8'(T',\|x\|)e^{nM_1T'} =: M_8(T',\|x\|) \end{aligned} \tag{123}$$

when $T' \geq \tau - t$. Here,

$$\begin{aligned} M_8'(T',\|x\|) = {} & n^3M_3 + n^6T'M_3M_6(T',\|x\|)M_4^2(T') \\ & + 2n^5T'M_2M_7(T',\|x\|)M_4(T') \\ & + n^5T'M_2M_6(T',\|x\|)M_5(T'). \end{aligned} \tag{124}$$

Appendix E. The Liapunov Function $V(x,t)$

By adding the absolute values of (84) for all $i,j \in \{1,\ldots,n\}$, one obtains that

$$\sum_{i,j=1}^{n}\left|\frac{\partial^2 V}{\partial x_i\,\partial x_j}(x,t)\right| \leq 2\int_t^{t+T_V} \sum_{i,j,m=1}^{n}\left|\frac{\partial^2\Phi_m}{\partial x_i\,\partial x_j}(\tau,t,x)\right| k\|x\| \tag{125}$$

$$+ \sum_{m=1}^{n}\left(\left(\sum_{i=1}^{n}\left|\frac{\partial\Phi_m}{\partial x_i}(\tau,t,x)\right|\right)\left(\sum_{j=1}^{n}\left|\frac{\partial\Phi_m}{\partial x_j}(\tau,t,x)\right|\right)\right) d\tau \tag{126}$$

since $|\Phi_m(\tau,t,x)| \le k\|x\| < kr_0$. By (106), $\sum_{j=1}^n |(\partial\Phi_m/\partial x_j)(\tau,t,x)| \le M_4(T_V)$ for all $m \in \{1,\ldots,n\}$ such that (110) and (125) imply that

$$\sum_{i,j=1}^n \left|\frac{\partial^2 V}{\partial x_i\,\partial x_j}(x,t)\right| \le 2T_V M_5(T_V)k\|x\| + 2\int_t^{t+T_V}\left(\sum_{m,i=1}^n \left|\frac{\partial\Phi_m}{\partial x_i}(\tau,t,x)\right|\right) M_4(T_V)\,d\tau$$
$$\le 2T_V(M_5(T_V)k\|x\| + M_4^2(T_V)) =: M_9(\|x\|). \tag{127}$$

By adding the absolute values of (91) for all $i,j \in \{1,\ldots,n\}$, one obtains by invoking the triangle inequality and using (113) that

$$\begin{aligned}
&\sum_{i,j=1}^n \left|\frac{\partial^3 V}{\partial x_i\,\partial x_j\,\partial t}(x,t)\right| \\
&\quad\le 2n + 2\sum_{i,j,m=1}^n \left|\frac{\partial^2\Phi_m}{\partial x_i\,\partial x_j}(t+T_V,t,x)\right| k\|x\| \\
&\qquad + 2\sum_{i,j,m=1}^n \left|\frac{\partial\Phi_m}{\partial x_i}(t+T_V,t,x)\frac{\partial\Phi_m}{\partial x_j}(t+T_V,t,x)\right| \\
&\qquad + 2\int_t^{t+T_V} \sum_{i,j,m=1}^n \left|\frac{\partial^3\Phi_m}{\partial x_i\,\partial x_j\,\partial t}(\tau,t,x)\right| k\|x\| \\
&\qquad + \sum_{i,j,m=1}^n \left|\frac{\partial^2\Phi_m}{\partial x_i\,\partial x_j}(\tau,t,x)\right| M_6(T_V,\|x\|)\,d\tau \\
&\qquad + 2\int_t^{t+T_V} 2\sum_{m=1}^n\left(\left(\sum_{i=1}^n\left|\frac{\partial^2\Phi_m}{\partial x_i\,\partial t}(\tau,t,x)\right|\right)\left(\sum_{j=1}^n\left|\frac{\partial\Phi_m}{\partial x_j}(\tau,t,x)\right|\right)\right) d\tau
\end{aligned} \tag{128}$$

since $|\Phi_m(\tau,t,x)| \le k\|x\| < kr_0$. By (110), it is clear that

$$\sum_{i,j,m=1}^n \left|\frac{\partial^2\Phi_m}{\partial x_i\,\partial x_j}(t+T_V,t,x)\right| \le M_5(T_V). \tag{129}$$

Relying on (106), it is clear that

$$\begin{aligned}
&\sum_{i,j,m=1}^n \left|\frac{\partial\Phi_m}{\partial x_i}(t+T_V,t,x)\frac{\partial\Phi_m}{\partial x_j}(t+T_V,t,x)\right| \\
&\quad= \sum_{m=1}^n\left(\left(\sum_{i=1}^n\left|\frac{\partial\Phi_m}{\partial x_i}(t+T_V,t,x)\right|\right)\left(\sum_{j=1}^n\left|\frac{\partial\Phi_m}{\partial x_j}(t+T_V,t,x)\right|\right)\right) \\
&\quad\le M_4^2(T_V).
\end{aligned} \tag{130}$$

Relying on (106) and (117), it is clear that for all $m \in \{1,\ldots,n\}$,

$$\sum_{j=1}^n \left|\frac{\partial\Phi_m}{\partial x_j}(\tau,t,x)\right| \le M_4(T_V) \quad\text{and}\quad \sum_{i=1}^n \left|\frac{\partial^2\Phi_m}{\partial x_i\,\partial t}(\tau,t,x)\right| \le M_7(T_V,\|x\|) \tag{131}$$

when $\tau \leq t + T_V$. Relying on (129), (130), (123), (110) and (131), it is clear that

$$\begin{aligned}\sum_{i,j=1}^{n} \left| \frac{\partial^3 V}{\partial x_i \, \partial x_j \, \partial t}(x,t) \right| &\\ &\leq 2(n + M_5(T_V)k\|x\| + M_4^2(T_V)) \\ &\quad + 2T_V(M_8(T_V, \|x\|)k\|x\| + M_5(T_V)M_6(T_V, \|x\|) \\ &\quad + 2M_4(T_V)M_7(T_V, \|x\|)) \\ &=: M_{10}(\|x\|). \end{aligned} \tag{132}$$

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