TABLE I
COMPARISON OF STABILITY RELATED MEASURES, ESTIMATED MINIMUM BIT LENGTHS AND TRUE MINIMUM BIT LENGTHS FOR THE INITIAL AND OPTIMAL CONTROLLER REALIZATIONS

realization	μ_1	$\hat{B}_{s1}^{\min}$	$B_s^{\min}$
initial $\mathbf{w}_0$	1.995885e-5	22	15
optimal $\hat{\mathbf{w}}_{\mathrm{opt}}$	6.019238e-4	14	7

minimum bit lengths for the initial and optimal controller realizations. It can be seen that, for this example, the optimization achieved an improvement by a factor of 30 on the closed-loop stability related measure and an 8-bit reduction in the required minimum bit length.

VI. CONCLUSIONS

In this paper, we have presented an approach to address the stability issues of the closed-loop discrete-time system where a state-estimate feedback controller is implemented with a fixed-point processor. An FWL closed-loop stability related measure has been derived, which is computationally tractable. As this measure is a function of the controller realization; the optimal realization problem of state-estimate feedback controllers is to find a realization that maximizes this measure. It has been shown that this optimal realization problem can be interpreted as a nonlinear programming problem. An efficient global optimization strategy based on the ASA algorithm has been adopted to solve this nonsmooth and nonconvex optimization problem.

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Practical Stability and Stabilization

Luc Moreau and Dirk Aeyels

***Abstract*—We present a practical stability result for dynamical systems depending on a small parameter. This result is applied to a practical stability analysis of fast time-varying systems studied in averaging theory, and of highly oscillatory systems studied by Sussmann and Liu. Furthermore, the problem of practically stabilizing control affine systems with drift is discussed.**

***Index Terms*—Approximation methods, Lie algebras, stability, time-varying systems.**

I. INTRODUCTION

In the present note, dynamical systems that depend on a small parameter are studied from the viewpoint of continuity of solutions.

Consider a system that depends on a small parameter $\varepsilon > 0$

$$\dot{x} = f^{\varepsilon}(t, x) \tag{1}$$

and a system

$$\dot{x} = g(t, x) \tag{2}$$

with the assumption that trajectories of (1) converge—uniformly on compact time intervals—to trajectories of (2) as $\varepsilon \downarrow 0$.

A particular example is given by fast time-varying systems studied in averaging theory

$$\dot{x} = f\left(\frac{t}{\varepsilon}, x\right). \tag{3}$$

It is well known that, under appropriate technical conditions, there exists an associated averaged system

$$\dot{x} = f_{\mathrm{av}}(x) \tag{4}$$

such that trajectories of (3) converge—uniformly on compact time intervals—to trajectories of (4) as $\varepsilon \downarrow 0$.

Teel *et al.* [1] have proven that, under appropriate technical conditions, if the origin of the averaged system (4) is a globally asymptotically stable equilibrium point, then the fast time-varying system (3) is practically stable. Their proof is based on advanced Lyapunov techniques.

In the present note, it is recognized that this practical stability result is of a topological nature, that it is a consequence of the convergence property of solutions: we prove the general result that, under appropriate technical conditions, if the origin of system (2) is a globally uniformly asymptotically stable equilibrium point, then system (1) is practically stable. This approach provides an alternative proof for the practical stability result [1] mentioned above, and extends it to a larger class of systems: it is not only applicable to fast time-varying systems as in averaging theory, but also, for example, to highly oscillatory systems studied by Sussmann and Liu [2]. This latter application is useful for control purposes. Indeed, it leads to a practical stabilization algorithm for a class of control affine systems with drift.

An outline of this note is as follows. Section II introduces some notations and hypotheses. Section III introduces a notion of practical

Manuscript received April 8, 1999; revised November 29, 1999. Recommended by Associate Editor, W. Lin. The work of L. Moreau was supported by BOF Grant 011D0696 of the Ghent University.

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Publisher Item Identifier S 0018-9286(00)06078-5.

stability and presents a practical stability theorem. Section IV is devoted to control applications. A preliminary version of this note has appeared as a conference paper [3].

II. PRELIMINARIES

The state space for all systems featuring in the present note is $\mathbb{R}^n$ with $n \in \mathbb{N}$. We consider two systems: a system that depends on a parameter $\varepsilon \in (0, \varepsilon_0]$ $(\varepsilon_0 \in (0, \infty))$

$$\dot{x} = f^\varepsilon(t, x) \tag{5}$$

and a system

$$\dot{x} = g(t, x). \tag{6}$$

We make the following hypothesis:

Hypothesis 1: (Existence and uniqueness conditions) 1) For each ε, $f^\varepsilon: \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$ is continuous and $f^\varepsilon(t, \cdot): \mathbb{R}^n \to \mathbb{R}^n$ is locally Lipschitz uniformly with respect to t for t belonging to compact time intervals. 2) $g: \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$ is continuous and $g(t, \cdot): \mathbb{R}^n \to \mathbb{R}^n$ is locally Lipschitz uniformly with respect to t for t belonging to compact time intervals.

This hypothesis implies that systems (5) and (6) have the local existence and uniqueness property of trajectories. We do not assume forward completeness of solutions; that is, we do not exclude finite escape times.

Let $\phi^\varepsilon(t, t_0, x_0)$ be the trajectory of (5) passing through state x_0 at time t_0 evaluated at time t. The function $(t, t_0, x_0) \mapsto \phi^\varepsilon(t, t_0, x_0)$ is called the *flow* of this system. By Hypothesis 1-1), the domain of ϕ^ε is open and ϕ^ε is continuous on its domain for each ε; see [4, Appendix C] and [5, p. 94]. Similarly, the flow of (6) is defined as the function $(t, t_0, x_0) \mapsto \psi(t, t_0, x_0)$ with $\psi(t, t_0, x_0)$ the trajectory of (6) passing through state x_0 at time t_0 evaluated at time t. The domain of ψ is open and ψ is continuous on its domain by Hypothesis 1-2).

Throughout the note, we assume that trajectories of (5) converge to those of (6) in the following sense:

Hypothesis 2: (Convergence of trajectories) For every $T \in (0, \infty)$ and compact set $K \subset \mathbb{R}^n$ satisfying $\{(t, t_0, x_0) \in \mathbb{R} \times \mathbb{R} \times \mathbb{R}^n : t \in [t_0, t_0 + T],\ x_0 \in K\} \subset \operatorname{Dom} \psi$, for every $d \in (0, \infty)$, there exists $\varepsilon^* \in (0, \varepsilon_0]$ such that for all $t_0 \in \mathbb{R}$, for all $x_0 \in K$ and for all $\varepsilon \in (0, \varepsilon^*)$

$$\begin{cases} \phi^\varepsilon(t, t_0, x_0) \text{ exists} \\ \|\phi^\varepsilon(t, t_0, x_0) - \psi(t, t_0, x_0)\| < d \end{cases} \quad \forall t \in [t_0, t_0 + T]. \tag{7}$$

In other words, we require that trajectories of (5) converge uniformly on compact time intervals to trajectories of (6) as $\varepsilon \downarrow 0$, and furthermore, we assume that this convergence is uniform with respect to t_0 and x_0 for $t_0 \in \mathbb{R}$ and x_0 belonging to compact sets. It is important to notice that the assumed convergence is not stated in terms of vector-fields, but in terms of trajectories; we do not assume that f^ε converges pointwise to g as $\varepsilon \downarrow 0$.

Example 1: (Fast time-varying systems) Given functions $f: \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n: (t, x) \mapsto f(t, x)$ and $f_{\mathrm{av}}: \mathbb{R}^n \to \mathbb{R}^n: x \mapsto f_{\mathrm{av}}(x)$ that are assumed to satisfy the following conditions: i) f is continuous, $f(t, \cdot): \mathbb{R}^n \to \mathbb{R}^n$ is locally Lipschitz uniformly with respect to t for $t \in \mathbb{R}$, and $f(\cdot, x): \mathbb{R} \to \mathbb{R}^n$ is bounded uniformly with respect to x for x in compact subsets of $\mathbb{R}^n$; ii) f_{av} is locally Lipschitz; and iii) for each compact set $K \subset \mathbb{R}^n$ and each $T \in (0, \infty)$

$$\int_{t_0}^{t_0+\theta} \left\{ f\left(\frac{s}{\varepsilon}, x\right) - f_{\mathrm{av}}(x) \right\} \mathrm{d}s \to 0 \tag{8}$$

as $\varepsilon \downarrow 0$ uniformly with respect to t_0, θ and x for $t_0 \in \mathbb{R}$, $\theta \in [0, T]$ and $x \in K$. System

$$\dot{x} = f\left(\frac{t}{\varepsilon}, x\right) \tag{9}$$

is called a fast time-varying system, and

$$\dot{x} = f_{\mathrm{av}}(x) \tag{10}$$

the associated averaged system. Systems (9) and (10) satisfy Hypothesis 1 by assumption and Hypothesis 2—this may be proven based on the Gronwall Lemma similar as in [6]; slightly different convergence results may be found, for example, in [7]. Consequently, all results obtained in the general framework of the present note apply in particular to fast time-varying systems (9) and their averaged (10).

Example 2: (Highly oscillatory systems) Given vector fields $X_i: \mathbb{R}^n \to \mathbb{R}^n: x \mapsto X_i(x)$ $(i \in \{1, 2, 3\})$ of class C^2. System

$$\dot{x} = X_1(x) + \frac{1}{\sqrt{\varepsilon}} \cos\left(\frac{t}{\varepsilon}\right) X_2(x) + \frac{1}{\sqrt{\varepsilon}} \sin\left(\frac{t}{\varepsilon}\right) X_3(x) \tag{11}$$

is called a highly oscillatory system, and

$$\dot{x} = X_1(x) + \tfrac{1}{2}[X_2, X_3](x) \tag{12}$$

the associated extended system [2]. Systems (11) and (12) satisfy Hypothesis 1 by assumption and Hypothesis 2—this may be proven based on partial integration and the Gronwall Lemma similar as in [6]; slightly different convergence results may be found, for example, in [8] and [2]. Consequently, all results obtained in the general framework of the present note apply in particular to highly oscillatory systems (11) and their extended system (12).

III. PRACTICAL STABILITY

This section contains the main theorem of the note: under Hypotheses 1 and 2, global uniform asymptotic stability for (6) implies practical stability for (5).

Before we proceed, we recall the definition of global uniform asymptotic stability and we introduce the notion of practical global uniform asymptotic stability.

Definition 1: Consider system (6). Assume that Hypothesis 1-2) is satisfied and let ψ denote the flow of this system. Assume that the origin is an equilibrium point. This equilibrium point is called *globally uniformly asymptotically stable* (GUAS) if the following three conditions are all satisfied:

1) *Uniform Stability:* For every $c_2 \in (0, \infty)$, there exists $c_1 \in (0, \infty)$ such that for all $t_0 \in \mathbb{R}$ and for all $x_0 \in \mathbb{R}^n$ with $\|x_0\| < c_1$

$$\begin{cases} \psi(t, t_0, x_0) \text{ exists} & \forall t \in [t_0, \infty) \\ \|\psi(t, t_0, x_0)\| < c_2 & \forall t \in [t_0, \infty). \end{cases} \tag{13}$$

2) *Uniform Boundedness:* For every $c_1 \in (0, \infty)$, there exists $c_2 \in (0, \infty)$ such that for all $t_0 \in \mathbb{R}$ and for all $x_0 \in \mathbb{R}^n$ with $\|x_0\| < c_1$

$$\begin{cases} \psi(t, t_0, x_0) \text{ exists} & \forall t \in [t_0, \infty) \\ \|\psi(t, t_0, x_0)\| < c_2 & \forall t \in [t_0, \infty). \end{cases} \tag{14}$$

3) *Global Uniform Attractivity:* For all $c_1, c_2 \in (0, \infty)$, there exists $T \in (0, \infty)$ such that for all $t_0 \in \mathbb{R}$ and for all $x_0 \in \mathbb{R}^n$ with $\|x_0\| < c_1$

$$\begin{cases} \psi(t, t_0, x_0) \text{ exists} & \forall t \in [t_0, \infty) \\ \|\psi(t, t_0, x_0)\| < c_2 & \forall t \in [t_0 + T, \infty). \end{cases}$$

Remark 1: Condition 2 has to be included explicitly in Definition 1, it is not a consequence of conditions 1 and 3; see, for example, [9].

There exists an equivalent characterization of GUAS by means of class $\mathcal{KL}$ functions; see, for example, [10].

Definition 2: Consider system (5). Assume that Hypothesis 1-1) is satisfied and let ϕ^ε denote the flow of this system. We call the origin of this system *practically globally uniformly asymptotically stable* (PGUAS) if the following three conditions are all satisfied:

1) For every $c_2 \in (0, \infty)$, there exist $c_1 \in (0, \infty)$ and $\hat{\varepsilon} \in (0, \varepsilon_0]$ such that for all $t_0 \in \mathbb{R}$, for all $x_0 \in \mathbb{R}^n$ with $\|x_0\| < c_1$ and for all $\varepsilon \in (0, \hat{\varepsilon})$

$$\begin{cases} \phi^\varepsilon(t, t_0, x_0) \text{ exists} & \forall\, t \in [t_0, \infty) \\ \|\phi^\varepsilon(t, t_0, x_0)\| < c_2 & \forall\, t \in [t_0, \infty). \end{cases} \tag{15}$$

2) For every $c_1 \in (0, \infty)$, there exist $c_2 \in (0, \infty)$ and $\hat{\varepsilon} \in (0, \varepsilon_0]$ such that for all $t_0 \in \mathbb{R}$, for all $x_0 \in \mathbb{R}^n$ with $\|x_0\| < c_1$ and for all $\varepsilon \in (0, \hat{\varepsilon})$

$$\begin{cases} \phi^\varepsilon(t, t_0, x_0) \text{ exists} & \forall\, t \in [t_0, \infty) \\ \|\phi^\varepsilon(t, t_0, x_0)\| < c_2 & \forall\, t \in [t_0, \infty). \end{cases} \tag{16}$$

3) For every $c_1, c_2 \in (0, \infty)$, there exist $T \in (0, \infty)$ and $\hat{\varepsilon} \in (0, \varepsilon_0]$ such that for all $t_0 \in \mathbb{R}$, for all $x_0 \in \mathbb{R}^n$ with $\|x_0\| < c_1$ and for all $\varepsilon \in (0, \hat{\varepsilon})$

$$\begin{cases} \phi^\varepsilon(t, t_0, x_0) \text{ exists} & \forall\, t \in [t_0, \infty) \\ \|\phi^\varepsilon(t, t_0, x_0)\| < c_2 & \forall\, t \in [t_0 + T, \infty). \end{cases} \tag{17}$$

It is instructive to have a closer look at the strong similarities between Definition 1 and Definition 2. The notion of PGUAS may be interpreted as follows. Condition 1 of Definition 2 defines a practical version of uniform stability. Condition 2 defines a practical version of uniform boundedness. Condition 3 captures a practical notion of global uniform attractivity: all trajectories starting in an arbitrarily large ball centered at the origin end up in an arbitrarily small ball centered at the origin for appropriate—depending on the radii of the considered balls—values of the parameter ε. Notice that the origin is not required to be an equilibrium point in Definition 2, nor that the flow be forward complete.

Remark 2: The notion of PGUAS introduced here—see also the preliminary version of this note [3]—coincides with the notion of semiglobal practical asymptotic stability from [1].

Consider again systems (5) and (6) introduced above satisfying Hypotheses 1 and 2. Assume that the origin is a GUAS equilibrium point of (6). It is well known that this does not imply that the origin is a GUAS equilibrium point of (5) even if ε is small. It seems however reasonable to expect that (5) inherits some weaker notion of stability. In Definition 2 we have introduced a weaker notion of stability: PGUAS. The following theorem asserts that this weaker stability property is indeed inherited by (5) if the origin is a GUAS equilibrium point of (6).

Theorem 1: (Practical stability) Given systems (5) and (6) satisfying Hypotheses 1 and 2. If the origin is a GUAS equilibrium point of (6), the origin of (5) is PGUAS.

Before we proceed with the proof, we briefly discuss this result. First, this theorem is relevant for a robustness analysis of control systems with respect to general perturbations that leave trajectories close to those of the idealized model. Roughly speaking, Theorem 1 says that GUAS for the studied idealization implies PGUAS in practice, provided that the unmodeled perturbations are such that they leave trajectories close to those of the idealized model. This interpretation also justifies the terminology "practical global uniform asymptotic stability." Second, Theorem 1 leads to a practical stabilization paradigm: it says that control systems may be practically stabilized by constructing feedback laws depending on a parameter ε in such a way that trajectories of the closed-loop system converge—uniformly on compact time intervals—to trajectories of a globally uniformly asymptotically stable system as $\varepsilon \downarrow 0$.

Proof: First of all, notice that the flow ψ is forward complete by the assumed GUAS property. We successively prove that conditions 1, 2, and 3 of Definition 2 are satisfied.

1) Take an arbitrary $c_2 \in (0, \infty)$ and let $b_2 \in (0, c_2)$. By the GUAS property of ψ—in particular, by uniform stability—there exists $c_1 \in (0, \infty)$ such that

$$\|\psi(t, t_0, x_0)\| < b_2 \qquad \forall\, t \in [t_0, \infty),\ \forall\, t_0 \in \mathbb{R},\ \forall\, x_0 \in \mathbb{R}^n \text{ with } \|x_0\| < c_1. \tag{18}$$

Let $b_1 \in (0, c_1)$. Since the equilibrium point $x = 0$ of ψ is globally uniformly attractive, there exists $T \in (0, \infty)$ such that

$$\|\psi(t, t_0, x_0)\| < b_1 \qquad \forall\, t \in [t_0 + T, \infty),\ \forall\, t_0 \in \mathbb{R},\ \forall\, x_0 \in \mathbb{R}^n \text{ with } \|x_0\| < c_1. \tag{19}$$

At this stage of the proof, we have estimates (18) and (19) for ψ with $0 < b_1 < c_1$, $0 < b_2 < c_2$, and $T > 0$.

Let $d = \min\{c_1 - b_1, c_2 - b_2\}$. Invoking Hypothesis 2—with $K = \{x \in \mathbb{R}^n \colon \|x\| \le c_1\}$—yields the existence of $\hat{\varepsilon} \in (0, \varepsilon_0]$ such that

$$\begin{cases} \phi^\varepsilon(t, t_0, x_0) \text{ exists} \\ \|\phi^\varepsilon(t, t_0, x_0) - \psi(t, t_0, x_0)\| < d \end{cases} \forall\, t \in [t_0, t_0 + T] \quad \forall\, t_0 \in \mathbb{R},\ \forall\, x_0 \in \mathbb{R}^n \text{ with } \|x_0\| \le c_1,\ \forall\, \varepsilon \in (0, \hat{\varepsilon}). \tag{20}$$

Estimates (18)–(20) together yield

$$\begin{cases} \phi^\varepsilon(t, t_0, x_0) \text{ exists} & \forall\, t \in [t_0, t_0 + T] \\ \|\phi^\varepsilon(t, t_0, x_0)\| < c_2 & \forall\, t \in [t_0, t_0 + T] \\ \|\phi^\varepsilon(t, t_0, x_0)\| < c_1 & \text{for } t = t_0 + T \end{cases} \quad \forall\, t_0 \in \mathbb{R},\ \forall\, x_0 \in \mathbb{R}^n \text{ with } \|x_0\| < c_1,\ \forall\, \varepsilon \in (0, \hat{\varepsilon}). \tag{21}$$

Since $\|\phi^\varepsilon(t_0 + T, t_0, x_0)\| < c_1$, a repeated application of (21) yields

$$\begin{cases} \phi^\varepsilon(t, t_0, x_0) \text{ exists} \\ \|\phi^\varepsilon(t, t_0, x_0)\| < c_2 \end{cases} \forall\, t \in [t_0, \infty) \quad \forall\, t_0 \in \mathbb{R},\ \forall\, x_0 \in \mathbb{R}^n \text{ with } \|x_0\| < c_1,\ \forall\, \varepsilon \in (0, \hat{\varepsilon}) \tag{22}$$

which is the property we had to prove.

2) Take an arbitrary $c_1 \in (0, \infty)$ and let $b_1 \in (0, c_1)$. By the GUAS property of ψ—in particular, by uniform boundedness and global uniform attractivity—there exist $b_2 \in (0, \infty)$ and $T \in (0, \infty)$ such that

$$\begin{cases} \|\psi(t, t_0, x_0)\| < b_2 & \forall\, t \in [t_0, \infty) \\ \|\psi(t, t_0, x_0)\| < b_1 & \forall\, t \in [t_0 + T, \infty) \end{cases} \quad \forall\, t_0 \in \mathbb{R},\ \forall\, x_0 \in \mathbb{R}^n \text{ with } \|x_0\| < c_1. \tag{23}$$

Let $c_2 \in (b_2, \infty)$. At this stage of the proof, we have estimate (23) for ψ with $0 < b_1 < c_1$, $0 < b_2 < c_2$, and $T > 0$, which is identical to the situation encountered in the proof of condition 1. Repeating the same argument as there yields the existence of $\hat{\varepsilon} \in (0, \varepsilon_0]$ such that

$$\begin{cases} \phi^\varepsilon(t, t_0, x_0) \text{ exists} \\ \|\phi^\varepsilon(t, t_0, x_0)\| < c_2 \end{cases} \forall\, t \in [t_0, \infty) \quad \forall\, t_0 \in \mathbb{R},\ \forall\, x_0 \in \mathbb{R}^n \text{ with } \|x_0\| < c_1,\ \forall\, \varepsilon \in (0, \hat{\varepsilon}) \tag{24}$$

which is the property we had to prove.

3) Take arbitrary $c_1, c_2 \in (0, \infty)$. By practical uniform stability—condition 1 of Definition 2—proven above, there exist $c_3 \in (0, \infty)$ and $\varepsilon^* \in (0, \varepsilon_0]$ such that

$$\begin{cases} \phi^\varepsilon(t, t_0, x_0) \text{ exists} \\ \|\phi^\varepsilon(t, t_0, x_0)\| < c_2 \end{cases} \forall\, t \in [t_0, \infty) \quad \forall\, t_0 \in \mathbb{R},\ \forall\, x_0 \in \mathbb{R}^n \text{ with } \|x_0\| < c_3,\ \forall\, \varepsilon \in (0, \varepsilon^*). \tag{25}$$

Let $b_3 \in (0,\ c_3)$. Since the equilibrium point $x = 0$ of ψ is globally uniformly attractive, there exists $T \in (0, \infty)$ such that

$$\|\psi(t, t_0, x_0)\| < b_3 \qquad \forall t \in [t_0 + T, \infty),\ \forall t_0 \in \mathbb{R}, \\ \forall x_0 \in \mathbb{R}^n \text{ with } \|x_0\| < c_1. \tag{26}$$

Let $d = c_3 - b_3$. Invoking Hypothesis 2—with $K = \{x \in \mathbb{R}^n\colon \|x\| \le c_1\}$—yields the existence of $\varepsilon^{\#} \in (0, \varepsilon_0]$ such that

$$\begin{cases} \phi^{\varepsilon}(t, t_0, x_0) \text{ exists} \\ \|\phi^{\varepsilon}(t, t_0, x_0) - \psi(t, t_0, x_0)\| < d \end{cases} \quad \forall t \in [t_0, t_0 + T] \\ \forall t_0 \in \mathbb{R},\ \forall x_0 \in \mathbb{R}^n \text{ with } \|x_0\| \le c_1,\ \forall \varepsilon \in (0, \varepsilon^{\#}). \tag{27}$$

Estimates (26) and (27) yield

$$\begin{cases} \phi^{\varepsilon}(t, t_0, x_0) \text{ exists} & \forall t \in [t_0, t_0 + T], \\ \|\phi^{\varepsilon}(t, t_0, x_0)\| < c_3 & \text{for } t = t_0 + T, \end{cases} \\ \forall t_0 \in \mathbb{R},\ \forall x_0 \in \mathbb{R}^n \text{ with } \|x_0\| < c_1,\ \forall \varepsilon \in (0, \varepsilon^{\#}). \tag{28}$$

This, together with (25), leads to

$$\begin{cases} \phi^{\varepsilon}(t, t_0, x_0) \text{ exists} & \forall t \in [t_0, \infty) \\ \|\phi^{\varepsilon}(t, t_0, x_0)\| < c_2 & \forall t \in [t_0 + T, \infty) \end{cases} \\ \forall t_0 \in \mathbb{R},\ \forall x_0 \in \mathbb{R}^n \text{ with } \|x_0\| < c_1,\ \forall \varepsilon \in (0, \hat{\varepsilon}) \tag{29}$$

where $\hat{\varepsilon} = \min\{\varepsilon^*, \varepsilon^{\#}\}$. This is the last property we had to prove. ■

Remark 3: The proof of Theorem 1 is based on an analysis of the flows ϕ^{ε} and ψ, making use of Hypothesis 2. We are therefore inclined to believe that the present approach lends itself naturally to generalizations, where the differential equations (5) and (6) do not necessarily satisfy the technical Hypothesis 1, or even where the flows ϕ^{ε} and ψ do not necessarily model systems described by differential equations.

Example 3: (Fast time-varying systems) Consider again the fast time-varying system (9) and its averaged (10) from Example 1 that are assumed to satisfy the assumptions introduced there. An application of Theorem 1 yields: if the origin is a GUAS equilibrium point of the averaged system (10), then the origin of the fast time-varying system (9) is PGUAS and thus, in particular, trajectories of (9) starting in an arbitrarily large ball centered at the origin end up in an arbitrarily small ball centered at the origin provided system (9) is sufficiently—depending on the radii of the considered balls—fast time-varying; that is, provided ε is sufficiently small. As mentioned in the Section I, this result has been proven in [1] by means of advanced Lyapunov techniques.

Example 4: (Highly oscillatory systems) Consider again the highly oscillatory system (11) and its extended system (12) from Example 2 that are assumed to satisfy the assumptions introduced there. An application of Theorem 1 yields: if the origin is a GUAS equilibrium point of the extended system (12), then the origin of the highly oscillatory system (11) is PGUAS and thus, in particular, trajectories of (11) starting in an arbitrarily large ball centered at the origin end up in an arbitrarily small ball centered at the origin provided system (11) is sufficiently—depending on the radii of the considered balls—highly oscillatory; that is, provided ε is sufficiently small.

We end this section with some remarks on exponential stability: it turns out that Theorem 1 is also useful for exponential stability results. Indeed, if system $\dot{x} = f^{\varepsilon}(t, x)$ is linear in the state variable, then PGUAS actually implies global uniform exponential stability for ε sufficiently small. Results in this direction may be found in [11] and [12]. Furthermore, if system $\dot{x} = f^{\varepsilon}(t, x)$ is a nonlinear system with equilibrium point at the origin that satisfies some additional hypotheses such that the linearization principle is applicable, then PGUAS for the linearization at the origin implies global uniform exponential stability for this linearization for ε sufficiently small; and this implies local uniform exponential stability of the null solution of the original nonlinear system $\dot{x} = f^{\varepsilon}(t, x)$ for ε sufficiently small.

IV. PRACTICAL STABILIZATION

In Section III, we have analyzed stability properties of dynamical systems depending on a small parameter. The present section is devoted to control applications.

Consider a control affine system on $\mathbb{R}^3$ with drift

$$\dot{x} = X_0(x) + u_1 X_1(x) + u_2 X_2(x) \tag{30}$$

with $x \in \mathbb{R}^3$ and $u_1, u_2 \in \mathbb{R}$. It is assumed that i) X_0, X_1, and X_2 are smooth—that is, of class C^{∞}—functions from $\mathbb{R}^3$ to $\mathbb{R}^3$; and that ii) $X_1(x)$, $X_2(x)$, and $[X_1, X_2](x)$ span $\mathbb{R}^3$ for all $x \in \mathbb{R}^3$.

A standard problem in control theory is the feedback stabilization problem, where one wants to find a feedback law such that the origin of the resulting closed-loop system has some desired stability properties. Consider the case that there does not exist $u_1, u_2 \in \mathbb{R}$ such that $X_0(0) + u_1 X_1(0) + u_2 X_2(0) = 0$. In this case, it is clearly impossible to find a continuous feedback law such that the resulting closed-loop system has an equilibrium point at the origin. And thus it is *a fortiori* impossible to asymptotically stabilize the origin by means of continuous feedback. Nevertheless, one may be interested in keeping the state x close to the ideal state $x = 0$. We are therefore led to the following practical stabilization problem:

Problem 1: For some $\varepsilon_0 \in (0, \infty)$, find smooth functions $u_i^{\varepsilon}\colon \mathbb{R} \times \mathbb{R}^3 \to \mathbb{R}\colon (t, x) \mapsto u_i^{\varepsilon}(t, x)$ $(i \in \{1, 2\}, \varepsilon \in (0, \varepsilon_0])$ such that the origin of (30) is PGUAS—as defined in Definition 2.

We present a solution to this problem based on Examples 2 and 4, incorporating ideas from [13, p. 1363] and [14]. The proposed solution makes systematic use of Lie brackets of vectorfields and Lie algebraic properties.

We propose a feedback law of the following form:

$$u_1^{\varepsilon}(t, x) = l_1(x) + \frac{1}{\sqrt{\varepsilon}} \cos\left(\frac{t}{\varepsilon}\right) l_3(x) \tag{31}$$

$$u_2^{\varepsilon}(t, x) = l_2(x) + \frac{1}{\sqrt{\varepsilon}} \sin\left(\frac{t}{\varepsilon}\right) \tag{32}$$

with smooth functions $l_i\colon \mathbb{R}^3 \to \mathbb{R}\colon x \mapsto l_i(x)$ $(i \in \{1, 2, 3\})$. With this choice of feedback, the closed-loop system becomes

$$\dot{x} = X_0(x) + l_1 X_1(x) + l_2 X_2(x) \\ + \frac{1}{\sqrt{\varepsilon}} \cos\left(\frac{t}{\varepsilon}\right) l_3 X_1(x) + \frac{1}{\sqrt{\varepsilon}} \sin\left(\frac{t}{\varepsilon}\right) X_2(x) \tag{33}$$

which is a highly oscillatory system with associated extended system—see Example 2—

$$\dot{x} = X_0(x) + l_1 X_1(x) + l_2 X_2(x) + \tfrac{1}{2}[l_3 X_1, X_2](x). \tag{34}$$

The practical stabilization problem is solved if the functions l_i can be chosen in such a way that the extended system (34) has a GUAS equilibrium point at the origin. Indeed, by Example 4, the origin of the closed-loop system (33) is PGUAS for this choice of functions l_i.

Based on Lie algebraic properties, (34) may be rewritten as

$$\dot{x} = X_0(x) + l_1 X_1(x) + l_2 X_2(x) - \tfrac{1}{2}(L_{X_2} l_3) X_1(x) \\ + \tfrac{1}{2} l_3 [X_1, X_2](x) \tag{35}$$

where $L_{X_2} l_3$ stands for the Lie derivative of l_3 along the vectorfield X_2.

Let $g\colon \mathbb{R}^3 \to \mathbb{R}^3\colon x \mapsto g(x)$ be a smooth function such that the origin is a GUAS equilibrium point of

$$\dot{x} = g(x). \tag{36}$$

By the span condition on X_1, X_2, and $[X_1, X_2]$, there exist smooth functions $k_i\colon \mathbb{R}^3 \to \mathbb{R}\colon x \mapsto k_i(x)$ such that

$$g(x) = X_0(x) + k_1 X_1(x) + k_2 X_2(x) + k_3[X_1, X_2](x) \qquad (37)$$

for all $x \in \mathbb{R}^3$.

By a judicious choice of the functions l_i we can make system (34) identical to system (36). Indeed, identifying the corresponding coefficients in the right-hand sides of (35) and (37) yields

$$l_1 = k_1 + \tfrac{1}{2} L_{X_2} l_3, \qquad l_2 = k_2,\ l_3 = 2k_3. \qquad (38)$$

For this choice of the functions l_i, system (34) has a GUAS equilibrium point at the origin, and hence, the origin of the closed-loop system (33) is PGUAS by Example 4.

We have thus solved the practical stabilization problem for a particular class of control affine systems with drift. Notice that the proposed method is constructive.

Remark 4: As mentioned above, in the case that there does not exist $u_1, u_2 \in \mathbb{R}$ such that $X_0(0) + u_1 X_1(0) + u_2 X_2(0) = 0$, it is natural to consider practical stabilization. However, if there does exist $u_1, u_2 \in \mathbb{R}$ such that $X_0(0) + u_1 X_1(0) + u_2 X_2(0) = 0$, then one can try to find an asymptotically stabilizing feedback law, and for the particular case that the drift vectorfield X_0 vanishes, Morin *et al.* [14] have actually reported an algorithm that yields locally uniformly exponentially—with respect to a *homogeneous norm*—stabilizing feedback laws.

V. CONCLUSION

We have introduced a notion of practical stability for dynamical systems depending on a small parameter. We have stated a practical stability theorem. We have applied this theory to a practical stability analysis of fast time-varying systems studied in averaging theory, and of highly oscillatory systems studied by Sussmann and Liu. We have used this theory for the practical stabilization of a class of control affine systems with drift.

ACKNOWLEDGMENT

One of the authors (L. Moreau) would like to thank P. Morin for a fruitful discussion on the subject of stabilizing driftless control affine systems.

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On Cone-Invariant Linear Matrix Inequalities

Pablo A. Parrilo and Sven Khatri

***Abstract*—An exact solution for a special class of cone-preserving linear matrix inequalities (LMIs) is developed. By using a generalized version of the classical Perron–Frobenius theorem, the optimal value is shown to be equal to the spectral radius of an associated linear operator. This allows for a much more efficient computation of the optimal solution using, for instance, power iteration-type algorithms. This particular LMI class appears in the computation of upper bounds for some generalizations of the structured singular value μ (spherical μ) and in a class of rank minimization problems previously studied. Examples and comparisons with existing techniques are provided.**

***Index Terms*—Linear matrix inequalities, Perron–Frobenius, structured singular value.**

I. INTRODUCTION

In the last few years, linear matrix inequalities (LMIs, see [1] for a comprehensive review) have become very useful tools in control theory. Numerous control-related problems, such as $\mathcal{H}_2$ and $\mathcal{H}_\infty$ analysis and synthesis, μ-analysis, model validation, etc., can be cast and solved in the LMI framework. LMI techniques not only have provided alternative (sometimes simpler) derivations of known results, but also supplied answers for previously unsolved problems.

LMIs are convex optimization problems that can be solved efficiently in polynomial time. The most effective computational approaches use projective or interior-point methods [2] to compute the optimal solutions.

However, for certain problems, the LMI formulation is not necessarily the most computationally efficient. A typical example of this is the computation of solutions of Riccati inequalities, appearing in $\mathcal{H}_\infty$

Manuscript received November 9, 1999; revised December 17, 1999. Recommended by Associate Editor, E. Feron.

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Publisher Item Identifier S 0018-9286(00)06753-2.