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Brief Paper

On exponential stability of nonlinear time-varying differential equations[1]

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Abstract

Within the Liapunov framework, a sufficient condition for exponential stability of ordinary differential equations is proposed. Unlike with classical Liapunov theory, the time derivative of the Liapunov function, taken along solutions of the system, may have positive and negative values. Verification of the conditions of the main theorem may be harder than in the classical case. It is shown that the proposed conditions are useful for the investigation of the exponential stability of fast time-varying systems. This sets the stability study by means of averaging in a Liapunov context. In particular, it is established that exponential stability of the averaged system implies exponential stability of the original fast time-varying system. A comparison of our work with results taken from the literature is included. © 1999 Elsevier Science Ltd. All rights reserved.

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1. Introduction

The classical Liapunov approach to uniform asymptotic stability of the null solution of a dynamical system $\dot{x}(t)=f(x,t)$ requires the existence of a positive definite, decrescent Liapunov function $V(x,t)$ whose derivative along the solutions of the system is negative definite. When this derivative is negative semi-definite, stability follows, rather than asymptotic stability. Narendra and Annaswamy (1987) show that with $\dot{V}(x,t)\le 0$ uniform asymptotic stability can be proven if there exists a T such that $\forall t$: $V(x(t+T),t+T)-V(x(t),t)\le -\gamma(\|x(t)\|)<0$. Here $\gamma(\cdot)$ is a strictly increasing continuous function on $\mathbb{R}^+$ which is zero at the origin, and $x(t+T)$ is the solution of the system at $t+T$ with initial condition $x(t)$ at t. Weaker conditions than the Narendra–Annaswamy conditions still implying asymptotic stability have recently been obtained, see (Aeyels, 1995; Aeyels and Sepulchre, 1994; Aeyels and Peuteman, 1998). The present paper and Aeyels and Peuteman (1998) are to a great extent inspired by the result of Narendra–Annaswamy (1987). In Aeyels and Peuteman (1998), we claim basically that in the uniform asymptotic stability theorem of Narendra–Annaswamy *the negative semi-definiteness condition on the time-derivative of the Liapunov function can be dispensed with.* Unlike Aeyels and Peuteman (1998), this paper focuses on exponential stability, not uniform asymptotic stability. We prove that also in the exponential stability case, no negative semi-definiteness condition on the time-derivative of the Liapunov function is required, i.e., the origin of a dynamical system is exponentially stable under the condition that for a positive definite decrescent $V(x,t)$ with $\lambda_{\min}\|x\|^2\le V(x,t)\le \lambda_{\max}\|x\|^2(\lambda_{\min},\lambda_{\max}\in\mathbb{R}_0^+)$, $\exists T>0$ and a sequence of times t_k^* such that $V(x(t_{k+1}^*),t_{k+1}^*)-V(x(t_k^*),t_k^*)\le -\nu\|x(t_k^*)\|^2$ with $\nu\in\mathbb{R}_0^+$ and $t_{k+1}^*-t_k^*\le T$ $\forall k\in\mathbb{Z}$ and $t_k^*\to\infty$ as $k\to\infty$ and $t_k^*\to-\infty$ as $k\to-\infty$. Compared to Aeyels and Peuteman (1998), the asymptotic stability condition needs to be satisfied only for a sequence t_k^*, not for all t. In comparison to Narendra and Annaswamy (1987), where a *global* uniform asymptotic stability result is proved, the present paper is mainly focused on *local* exponential stability. A global exponential stability result may be obtained when the system $\dot{x}=f(x,t)$ is globally Lipschitz. This main theorem is instrumental in proving that exponential stability of the

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averaged system $\dot{x} = \bar{f}(x)$ guarantees exponential stability of the original fast time-varying system, i.e., $\dot{x} = f(x, \alpha t)$ with α sufficiently large. The main lines of the argument are as follows. First, the converse theorem of Liapunov for exponential stability guarantees the existence of a Liapunov function $V(x)$ whose derivative along the flow of the averaged system is negative definite. Second, this $V(x)$ is used to establish exponential stability of $\dot{x} = f(x, \alpha t)$ for α sufficiently large. In general, the derivative of $V(x)$ along the flow of $\dot{x} = f(x, \alpha t)$ may take positive as well as negative values which precludes the use of a classical theorem of Liapunov. For α sufficiently large, we prove the existence of a $v > 0$, a $T > 0$ and a sequence of times t_k^*, satisfying the conditions mentioned above, i.e., $\forall k \in \mathbb{Z}$: $V(x(t_{k+1}^*)) - V(x(t_k^*)) \leq -v\|x(t_k^*)\|^2$. In Section 2 of this paper, the main theorem is established, giving new sufficient conditions which guarantee exponential stability of a nonlinear dynamical system. In Section 3, exponential stability of fast time-varying systems is discussed using the main theorem of Section 2. In Section 4, it is shown that local exponential stability of the averaged system guarantees local exponential stability of the original fast time-varying system. In the case the averaged system is globally exponentially stable and globally Lipschitz, a *semi-global* exponential stability result for $\dot{x} = f(x, \alpha t)$ is obtained, i.e., every bounded region of exponential stability for $\dot{x} = f(x, \alpha t)$ is obtained when α is sufficiently large. In Section 5, the results of this paper are compared with results taken from the literature.

2. The main theorem

In this section, sufficient conditions for exponential stability of a dynamical system are proposed. In classical Liapunov theory, the time derivative of the Liapunov function along the solutions of the system is required to be negative definite. In the present case, the derivative of the Liapunov function may have positive and negative values: a sequence of times t_k^* should exist such that the value of the Liapunov function decreases when evaluated at t_k^*. Consider

$$\dot{x}(t) = f(x(t), t) \tag{1}$$

with $f: W \times \mathbb{R} \to \mathbb{R}^n$, W open, $W \subset \mathbb{R}^n$, $f(x, t)$ is measurable for each fixed x, $0 \in W$ and $f(0, t) = 0$ $\forall t \in \mathbb{R}$. Furthermore, we assume that conditions are imposed on Eq. (1) such that existence and uniqueness of its solutions is secured. Of these conditions, we single out the local Lipschitz condition because it will be used in the course of the proofs of the theorems thereafter: f is locally Lipschitz on W, i.e., for $\forall x \in W$, $\exists$ a neighborhood $N(x) \subset W$, such that the restriction $f|_{N(x)}$ is Lipschitz with Lipschitz function $l_x(t)$. We assume that $l_x(t)$ is bounded $\forall t \in \mathbb{R}$. We are now ready to state the main theorem. For completeness, the following lemma is included.

Lemma 1. *Let $U \subset W$ be an open neighborhood of 0. Consider a closed ball $\bar{B}_\mu(0) \subset U$ then $\forall T > 0$ $\exists \mu' > 0$ such that $(x_0, t_0) \in B_{\mu'}(0) \times \mathbb{R}$ implies that $x(t) \in B_\mu(0)$ for $t \in [t_0, t_0 + T]$. Here $x(t)$ is the solution of Eq. (1) evaluated at t with initial condition x_0 at t_0. (The solution is assumed to exist over the considered time interval.)*

Proof. For a proof, the reader is referred to Aeyels and Peuteman (1998). □

Remark 1. The proof of the lemma shows that $\mu' = \mu e^{-KT}$ is an appropriate choice of μ'. K is the maximum of the upper bounds of the Lipschitz functions $l_x(t)$ corresponding to a finite subcovering of $\bar{B}_\mu(0)$ which consists of neighborhoods $N(x)$.

Theorem 1. *Consider a function $V: U \times \mathbb{R} \to \mathbb{R}$, with $U \subset W$ an open neighborhood of 0. We assume that the following conditions are satisfied:*

Condition 1: There exist strictly positive numbers $\lambda_{\min}$ and $\lambda_{\max}$ such that $\forall x \in U: \lambda_{\min}\|x\|^2 \leq V(x, t) \leq \lambda_{\max}\|x\|^2$ and $V(0, t) = 0$ $\forall t$.

Condition 2: There exists an increasing sequence of times t_k^* $(k \in \mathbb{Z})$ with $t_k^* \to \infty$ as $k \to \infty$ and $t_k^* \to -\infty$ as $k \to -\infty$, $\exists$ finite $T > 0: t_{k+1}^* - t_k^* \leq T$ $(\forall k \in \mathbb{Z})$, $\exists$ $v > 0$ and an open set $U' \subset U$ which contains the origin such that the Liapunov function $V(x(t), t)$ has the property that $\forall$ $k \in \mathbb{Z}$ and $\forall$ $x \in U' \backslash \{0\}$

$$V(x(t_{k+1}^*), t_{k+1}^*) - V(x, t_k^*) \leq -v\|x\|^2 < 0. \tag{2}$$

Here $x(t_{k+1}^*)$ is the solution of Eq. (1) at t_{k+1}^* with initial condition x at t_k^*.

Then the equilibrium point $x = 0$ of Eq. (1) is exponentially stable.

Proof. Local exponential stability means by definition that $\exists r > 0$, $\exists \gamma > 1$ and $\exists \lambda > 0$ such that $\forall$ $t_0 \in \mathbb{R}$, $\forall x_0$ with $\|x_0\| < r$, the corresponding solution $x(t)$ satisfies

$$\|x(t)\| \leq \gamma e^{-\lambda(t - t_0)}\|x(t_0)\|, \quad \forall t \geq t_0 \tag{3}$$

with $x(t_0) = x_0$. Consider a closed ball $\bar{B}_\varepsilon(0)$ centered at 0 and radius ε small enough such that $\bar{B}_\varepsilon(0) \subset U'$. Since $\bar{B}_\varepsilon(0)$ is compact, it can be covered with a finite number of open neighborhoods $N(x)$, $x \in \bar{B}_\varepsilon(0)$. Let K be the maximum of the upper bounds of the Lipschitz functions $l_x(t)$. Define

$$\varepsilon' := \varepsilon\, e^{-KT}, \qquad \delta' := \sqrt{\frac{\lambda_{\min}}{\lambda_{\max}}}\,\varepsilon' \quad \text{and} \quad \delta'' := e^{-KT}\delta'.$$

Consider the open balls $B_{\delta'}(0)$ and $B_{\delta''}(0)$. These definitions imply that $B_{\delta'}(0) \subset U' \subset U$ and therefore by

Lemma 1 and Remark 1 with $\mu = \delta'$ and $\mu' = \delta''$ one obtains that for all $(x_0, t_0) \in B_{\delta''}(0) \times \mathbb{R}$

$$\|x(t)\| \le e^{KT}\|x(t_0)\| < \delta' \quad t \in [t_0, t_0 + T]. \tag{4}$$

There exists a $k_0 \in \mathbb{Z}$ such that $t^*_{k_o} - t_0 < T$ implying that Eq. (4) holds for every $t \in [t_0, t^*_{k_o}]$ and in particular $\|x(t^*_{k_o})\| < \delta' \le \varepsilon' < \varepsilon$. By Condition 1, $V(x(t^*_{k_o}), t^*_{k_o}) \le \lambda_{\max}\|x(t^*_{k_o})\|^2$ and by Condition 2, $V(x(t^*_{k_o+1}), t^*_{k_o+1}) - V(x(t^*_{k_o}), t^*_{k_o}) \le -\nu\|x(t^*_{k_o})\|^2$ such that

$$V(x(t^*_{k_o+1}), t^*_{k_o+1}) \le \left(1 - \frac{\nu}{\lambda_{\max}}\right)V(x(t^*_{k_o}), t^*_{k_o}) \tag{5}$$

where $0 \le (1 - \nu/\lambda_{\max}) < 1$. Taking the definition of δ' into account, Eq. (5) and Condition 1 imply that $x(t^*_{k_o+1}) \in B_{\varepsilon'}(0)$. Therefore Conditions 1 and 2 can be applied and thus, by repeating this argument one obtains that $\forall n \in \mathbb{N}$

$$V(x(t^*_{k_o+n}), t^*_{k_o+n}) \le \left(1 - \frac{\nu}{\lambda_{\max}}\right)^n V(x(t^*_{k_o}), t^*_{k_o}). \tag{6}$$

Notice that by the same argument $\forall n \in \mathbb{N}$, $x(t^*_{k_o+n}) \in B_{\varepsilon'}(0)$. Take $n^* \in \mathbb{N}$ such that $(1 - \nu/\lambda_{\max})^{n^*} < \lambda_{\min}/\lambda_{\max}$. Since $\|x(t^*_{k_o+n^*})\| < \varepsilon'$, Eq. (6) implies by Condition 1 that

$$\lambda_{\min}\|x(t^*_{k_o+n^*})\|^2 \le \left(1 - \frac{\nu}{\lambda_{\max}}\right)^{n^*} \lambda_{\max}\|x(t^*_{k_o})\|^2. \tag{7}$$

Define $\nu^* := (1 - \nu/\lambda_{\max})^{n^*}\lambda_{\max}/\lambda_{\min} < 1$. By Eq. (7), $\|x(t^*_{k_o+n^*})\| \le \sqrt{\nu^*}\|x(t^*_{k_o})\|$. Define $\lambda := \ln(1/\nu^*)/2n^*T$ which implies that $\sqrt{\nu^*} = e^{-\lambda n^* T} \le e^{-\lambda(t^*_{k_o+n^*} - t^*_{k_o})}$ and

$$\|x(t^*_{k_o+n^*})\| \le e^{-\lambda(t^*_{k_o+n^*} - t^*_{k_o})}\|x(t^*_{k_o})\|. \tag{8}$$

Notice that $\|x(t^*_{k_o+n})\| < \varepsilon' \quad \forall n \in \mathbb{N}$ and therefore $\|x(t^*_{k_o+mn^*})\| < \varepsilon' \ \forall m \in \mathbb{N}$. By repeating the same argument, one obtains that for every $m \in \mathbb{N}$

$$\|x(t^*_{k_o+mn^*})\| \le e^{-\lambda(t^*_{k_o+mn^*} - t^*_{k_o})}\|x(t^*_{k_o})\|. \tag{9}$$

Notice that Eq. (9) gives the evolution of the flow at times $t^*_{k_o+mn^*}$, but Eq. (9) gives no information about the flow between the times $t^*_{k_o+mn^*}$. This will be examined now. Since $\forall n \in \mathbb{N}$: $\|x(t^*_{k_o+n})\| < \varepsilon'$, by Lemma 1 and Remark 1, one obtains that $\forall t \ge t^*_{k_o}$, $\|x(t)\| < \varepsilon$ with $B_\varepsilon(0) \subset U'$. For every $t \ge t^*_{k_o}$, there exists $m \in \mathbb{N}$ such that $t \in [t^*_{k_o+mn^*}, t^*_{k_o+(m+1)n^*}[$. Notice that

$$x(t) = x(t^*_{k_o+mn^*}) + \int_{t^*_{k_o+mn^*}}^{t} f(x(\tau), \tau)\,d\tau. \tag{10}$$

Since $\|f(x(\tau), \tau)\| \le K\|x(\tau)\|$, it is obvious by invoking the triangle inequality and the Bellman–Gronwall lemma that

$$\|x(t)\| \le e^{K(t - t^*_{k_o+mn^*})}\|x(t^*_{k_o+mn^*})\|. \tag{11}$$

By Eq. (9), $\|x(t)\| \le e^{-\lambda(t^*_{k_o+mn^*} - t^*_{k_o})}\, e^{K(t - t^*_{k_o+mn^*})}\|x(t^*_{k_o})\|$ and $\|x(t)\| \le e^{-\lambda(t - t^*_{k_o})}e^{(\lambda+K)n^*T}\|x(t^*_{k_o})\|$ if $t \ge t^*_{k_o}$. Since $\|x(t^*_{k_o})\| < \|x(t_0)\|e^{KT}$ and $t^*_{k_o} - t_0 < T$, one obtains that

$$\|x(t)\| \le e^{-\lambda(t - t_o)}e^{(\lambda+K)(n^*+1)T}\|x(t_0)\|, \quad t \ge t^*_{k_o}. \tag{12}$$

If $t \in [t_0, t^*_{k_o}[$ then by Eq. (4) one obtains that

$$\|x(t)\| \le e^{KT}\|x(t_0)\| \le e^{-\lambda(t - t_o)}e^{(\lambda+K)(n^*+1)T}\|x(t_0)\|.$$

In conclusion, we have established that $\exists r > 0$, $\exists$ finite $\gamma > 0$ and a finite $\lambda > 0$ such that $\forall t_0$, $\forall x_0$ with $\|x_0\| < r$, the corresponding solution $x(t)$ satisfies Eq. (3) by taking $r = \delta''$, $\gamma = e^{(\lambda+K)(n^*+1)T}$ and $\lambda = \ln(1/\nu^*)/2n^*T$ with $\nu^* = (1 - \nu/\lambda_{\max})^{n^*}\lambda_{\max}/\lambda_{\min}$. Therefore, exponential stability of the null solution of Eq. (1) is established. □

Remark 2. Theorem 1 not only establishes local exponential stability of the equilibrium point $x = 0$ of $\dot{x} = f(x, t)$, the proof of Theorem 1 also provides an estimate of the region of exponential stability. When $\bar{B}_\varepsilon(0) \subset U'$ then $B_{\delta''}(0)$ belongs to the region of exponential stability with $\delta'' = e^{-2KT}\sqrt{\lambda_{\min}/\lambda_{\max}}\,\varepsilon$.

When $\dot{x} = f(x, t)$ is globally Lipschitz, i.e., $\exists K > 0$ such that $\forall x, y \in \mathbb{R}^n$, $\forall t \in \mathbb{R}$: $\|f(x, t) - f(y, t)\| \le K\|x - y\|$ and the conditions of Theorem 1 hold globally, then the equilibrium point $x = 0$ is globally exponentially stable.

Remark 3. Theorem 1 may be generalized in the sense that the existence of a $m > 0$ such that $\forall x \in U$: $\lambda_{\min}\|x\|^m \le V(x, t) \le \lambda_{\max}\|x\|^m$ and $\forall x \in U'\setminus\{0\}$: $V(x(t^*_{k+1}), t^*_{k+1}) - V(x, t^*_k) \le -\nu\|x\|^m$ also implies exponential stability of the equilibrium point $x = 0$ of Eq. (1). This generalization allows the use of nonquadratic Liapunov functions.

3. Application of the main theorem to fast time-varying systems

Theorem 1 states a sufficient condition for exponential stability of a dynamical system. Condition 2 of Theorem 1 is in general hard to verify. This section can be seen as an elaboration of the previous one. We show that for fast time-varying systems, Condition 2 of Theorem 1 may be replaced by a condition independent of the flow, and therefore more easily verified. As explained in Section 4, Theorem 2 will be helpful in obtaining explicit conditions on the exponential stability of Eq. (13).

Theorem 2. *Consider the system*

$$\dot{x}(t) = f(x, \alpha t) \tag{13}$$

which satisfies the existence and uniqueness conditions mentioned in Section 2 with $f: W \times \mathbb{R} \to \mathbb{R}^n$, W open, $W \subset \mathbb{R}^n$, $0 \in W$ and $f(0, t) = 0\ \forall t \in \mathbb{R}$. There exists $\alpha_1 > 0$ such that $\forall \alpha > \alpha_1$ system (13) *has an exponentially stable equilibrium point $x = 0$ if*

Condition 1: *There exists a Liapunov function $V(x)$ and strictly positive numbers $\lambda_{\min}$ and $\lambda_{\max}$ such that $\forall x \in U$ ($U \subset W$ and U is open and $0 \in U$):*

$$\lambda_{\min}\|x\|^2 = \lambda_{\min}x^{\mathrm{T}}x \leq V(x) \leq \lambda_{\max}x^{\mathrm{T}}x = \lambda_{\max}\|x\|^2.$$

In addition, $\partial V/\partial x(0) = 0$ and $\partial V/\partial x(x)$ is Lipschitz on U with Lipschitz constant K_v.

Condition 2: *There exists an increasing sequence of times $t_k^\diamond$ ($k \in \mathbb{Z}$) with $t_k^\diamond \to \infty$ as $k \to \infty$ and $t_k^\diamond \to -\infty$ as $k \to -\infty$, $\exists$ finite $T^\diamond > 0$: $t_{k+1}^\diamond - t_k^\diamond \leq T^\diamond$ ($\forall k \in \mathbb{Z}$) and $\exists\, \upsilon > 0$ such that $\forall t_k^\diamond$ and $\forall x \in U$*

$$\frac{\partial V}{\partial x}(x)\int_{t_k^\diamond}^{t_{k+1}^\diamond} f(x,\tau)\,\mathrm{d}\tau \leq -\upsilon\|x\|^2. \tag{14}$$

Proof. Corresponding to the sequence of times $t_k^\diamond$ and to each $\alpha > 0$, define the sequence of times $t_k^* := t_k^\diamond/\alpha$. Since α is finite, $t_k^* \to \infty$ as $k \to \infty$ and $t_k^* \to -\infty$ as $k \to -\infty$. The sequence of times t_k^* satisfies Condition 2 of Theorem 1: $\forall k: t_{k+1}^* - t_k^* \leq T$ with $T = T^\diamond/\alpha$. Consider the Liapunov function $V(x)$. By defining $\Delta V(t_{k+1}^*, t_k^*) := V(x(t_{k+1}^*)) - V(x(t_k^*))$, one obtains that $\Delta V(t_{k+1}^*, t_k^*)$ equals

$$\int_{t_k^*}^{t_{k+1}^*} \dot{V}(x(t),t)\,\mathrm{d}t = \int_{t_k^*}^{t_{k+1}^*} \frac{\partial V}{\partial x}(x(t))\,f(x(t),\alpha t)\,\mathrm{d}t.$$

Here $x(t)$ is the solution of Eq. (13) evaluated at t with initial condition $x(t_k^*)$ at t_k^*. By Theorem 1, exponential stability of the equilibrium point of Eq. (13) is shown if $\exists v > 0$ and an open set U' containing the origin such that $\forall t_k^*$, $\forall x(t_k^*) \in U'\backslash\{0\}$:

$$\Delta V(t_{k+1}^*, t_k^*) \leq -v\|x(t_k^*)\|^2. \tag{15}$$

Consider a closed ball $\bar{B}_{\tilde\varepsilon}(0) \subset U$ centered at 0 and radius $\tilde\varepsilon$ small enough such that $\bar{B}_{\tilde\varepsilon}(0) \subset U$. Since $\bar{B}_{\tilde\varepsilon}(0)$ is compact, it can be covered with a finite number of open neighborhoods $N(x) \subset W$, $x \in \bar{B}_{\tilde\varepsilon}(0)$. Let K be the maximum of the upper bounds of the Lipschitz functions $l_x(t)$. The upper bound of $l_x(t)$ is the same for the systems $\dot{x}(t) = f(x,t)$ and $\dot{x}(t) = f(x,\alpha t)$. Define $\tilde\varepsilon' := \tilde\varepsilon\mathrm{e}^{-KT}$ with $T = T^\diamond/\alpha$. Consider the open ball $B_{\tilde\varepsilon'}(0)$. With Lemma 1 and Remark 1 one obtains that for $t \in [t_k^*, t_{k+1}^*]$, $x(t) \in B_{\tilde\varepsilon}(0)$ when $x(t_k^*) \in B_{\tilde\varepsilon'}(0)$. Notice that

$$x(t) = x(t_k^*) + \int_{t_k^*}^{t} f(x(s),\alpha s)\,\mathrm{d}s. \tag{16}$$

Since $\|f(x(s),\alpha s)\| \leq K\|x(s)\|$ when $s \in [t_k^*, t]$ and $t \in [t_k^*, t_{k+1}^*]$,

$$\|x(s)\| \leq \mathrm{e}^{K(s-t_k^*)}\|x(t_k^*)\| \leq \mathrm{e}^{KT^\diamond/\alpha}\|x(t_k^*)\| \tag{17}$$

and

$$\|x(t) - x(t_k^*)\| \leq K\frac{T^\diamond}{\alpha}\,\mathrm{e}^{KT^\diamond/\alpha}\|x(t_k^*)\|. \tag{18}$$

Because of the Lipschitz condition imposed on Eq. (13) one obtains that

$$\|f(x(t),\alpha t) - f(x(t_k^*),\alpha t)\| \leq K^2\,\frac{T^\diamond}{\alpha}\,\mathrm{e}^{KT^\diamond/\alpha}\|x(t_k^*)\|. \tag{19}$$

By assumption, $\partial V/\partial x(x)$ is Lipschitz on U with Lipschitz constant K_v. Since $x(t) \in B_{\tilde\varepsilon}(0) \subset U$ and $x(t_k^*) \in B_{\tilde\varepsilon'}(0) \subset U$, one obtains that

$$\left\|\frac{\partial V}{\partial x}(x(t)) - \frac{\partial V}{\partial x}(x(t_k^*))\right\| \leq K_v K\,\frac{T^\diamond}{\alpha}\,\mathrm{e}^{KT^\diamond/\alpha}\|x(t_k^*)\|. \tag{20}$$

Now, we are ready to prove that Eq. (15) is satisfied. Notice that $\Delta V(t_{k+1}^*, t_k^*)$ equals

$$\int_{t_k^*}^{t_{k+1}^*} \frac{\partial V}{\partial x}(x(t_k^*))\,f(x(t_k^*),\alpha t)\,\mathrm{d}t \tag{21}$$

$$+ \int_{t_k^*}^{t_{k+1}^*}\left(\frac{\partial V}{\partial x}(x(t)) - \frac{\partial V}{\partial x}(x(t_k^*))\right)f(x(t_k^*),\alpha t)\,\mathrm{d}t \tag{22}$$

$$+ \int_{t_k^*}^{t_{k+1}^*} \frac{\partial V}{\partial x}(x(t_k^*))\,(f(x(t),\alpha t) - f(x(t_k^*),\alpha t))\,\mathrm{d}t \tag{23}$$

$$+ \int_{t_k^*}^{t_{k+1}^*}\left(\frac{\partial V}{\partial x}(x(t)) - \frac{\partial V}{\partial x}(x(t_k^*))\right) \times (f(x(t),\alpha t) - f(x(t_k^*),\alpha t))\,\mathrm{d}t. \tag{24}$$

If $x(t_k^*) \in B_{\tilde\varepsilon'}(0)$ then $\|f(x(t_k^*),\alpha t)\| \leq K\|x(t_k^*)\|$ and $\|\partial V/\partial x(x(t_k^*))\| \leq K_v\|x(t_k^*)\|$ since $\partial V/\partial x(0) = 0$ and $f(0,t) = 0$. By Eqs. (19) and (20), one obtains that the absolute values of Eqs. (22) and (23) are less than or equal to

$$K_v K^2 \mathrm{e}^{KT^\diamond/\alpha}\left(\frac{T^\diamond}{\alpha}\right)^2\|x(t_k^*)\|^2. \tag{25}$$

By Eqs. (19) and (20), one obtains that the absolute value of Eq. (24) is less than or equal to

$$K_v K^3 \mathrm{e}^{2KT^\diamond/\alpha}\left(\frac{T^\diamond}{\alpha}\right)^3\|x(t_k^*)\|^2. \tag{26}$$

Therefore, combining Eqs. (25) and (26) one obtains with $L(\alpha) := 2K_vK^2\mathrm{e}^{KT^\diamond/\alpha}T^{\diamond 2} + K_vK^3\mathrm{e}^{2KT^\diamond/\alpha}T^{\diamond 3}/\alpha$ that $\Delta V(t_{k+1}^*, t_k^*)$ is less than or equal to

$$\int_{t_k^*}^{t_{k+1}^*} \frac{\partial V}{\partial x}(x(t_k^*))\,f(x(t_k^*),\alpha t)\,\mathrm{d}t + \frac{L(\alpha)}{\alpha^2}\,\|x(t_k^*)\|^2 \tag{27}$$

if $x(t_k^*) \in B_{\tilde\varepsilon'}(0)$. By Eq. (14),

$$\Delta V(t_{k+1}^*, t_k^*) \leq -\frac{\upsilon}{\alpha}\,\|x(t_k^*)\|^2 + \frac{L(\alpha)}{\alpha^2}\,\|x(t_k^*)\|^2. \tag{28}$$

This expression (28) is true for every $\alpha > 0$. When $\alpha > \alpha_1$ with α_1 such that $-\upsilon/\alpha_1 + L(\alpha_1)/\alpha_1^2 = 0$, Eq. (15) is satisfied with $v := \upsilon/\alpha - L(\alpha)/\alpha^2 > 0$. Therefore, exponential stability of Eq. (13) is established by applying Theorem 1 with $U' = B_{\tilde\varepsilon'}(0)$. $\square$

By specializing Theorem 2 to the linear case, one obtains that there exists $\alpha_1 > 0$ such that $\forall \alpha > \alpha_1$ the system $\dot{x}(t) = A(\alpha t)\, x(t)$, with $A(t)$ bounded and measurable, has an exponentially stable null solution if: there exists an increasing sequence of times $t_k^{\diamond} (k \in \mathbb{Z})$ with $t_k^{\diamond} \to \infty$ as $k \to \infty$ and $t_k^{\diamond} \to -\infty$ as $k \to -\infty$, $\exists$ finite $T^{\diamond} > 0 : t_{k+1}^{\diamond} - t_k^{\diamond} \le T^{\diamond}$ ($\forall k \in \mathbb{Z}$), there exists a positive definite matrix P and $\exists\, \upsilon > 0$ such that $\forall t_k^{\diamond}$

$$\left(\int_{t_k^{\diamond}}^{t_{k+1}^{\diamond}} A^{\mathrm{T}}(\tau)\, \mathrm{d}\tau\right) P + P\left(\int_{t_k^{\diamond}}^{t_{k+1}^{\diamond}} A(\tau)\, \mathrm{d}\tau\right) \le -\upsilon I. \tag{29}$$

When the linear system is periodic, i.e., $A(t)$ has a period $T^{\diamond}$, then the result of Brockett (1970) is recovered by taking an equidistant sequence of times $t_k^{\diamond}$ such that $\forall k$: $t_{k+1}^{\diamond} - t_k^{\diamond} = T^{\diamond}$. Therefore condition (29) is equivalent with the requirement that $\int_{t_k^{\diamond}}^{t_{k+1}^{\diamond}} A(\tau)\, \mathrm{d}\tau$ is Hurwitz, i.e., the eigenvalues are in the open left half plane.

Remark 4. The proof of Theorem 2 provides an estimate of α_1. Indeed, taking the specific expression of $L(\alpha)$ into account, $-\upsilon/\alpha_1 + L(\alpha_1)/\alpha_1^2 = 0$ may be rewritten as

$$\mathrm{e}^{KT^{\diamond}/\alpha_1} \frac{T^{\diamond}}{\alpha_1} = \frac{1}{K}\left(-1 + \sqrt{1 + \frac{v}{K_v K T^{\diamond}}}\right). \tag{30}$$

Since for every $K > 0$ and for every $T^{\diamond} > 0$, $\mathrm{e}(\alpha_1) := \mathrm{e}^{KT^{\diamond}/\alpha_1}(T^{\diamond}/\alpha_1)$ is a monotonically decreasing function of α_1, since $e(\alpha_1)$ goes to infinity as α_1 goes to zero and since $e(\alpha_1)$ goes to zero as α_1 goes to infinity, Eq. (30) has for every $K > 0$, for every $K_v > 0$, for every $v > 0$ and for every $T^{\diamond} > 0$ exactly one solution α_1.

Remark 5. Theorem 2 also provides an estimate of the region of exponential stability. Because of Remark 2 and the proof of Theorem 2, one obtains that $B_{\delta''}(0)$ belongs to the region of exponential stability of $\dot{x} = f(x, \alpha t)$ when $\delta'' < \mathrm{e}^{-3KT^{\diamond}/\alpha}\sqrt{\lambda_{\min}/\lambda_{\max}}\, \tilde{\varepsilon}$, $\bar{B}_{\tilde{\varepsilon}}(0) \subset U$ and $\alpha > \alpha_1$. When $f(x, t)$ and $(\partial V/\partial x)(x)$ are globally Lipschitz and the conditions of Theorem 2 hold globally, then the equilibrium point $x = 0$ is globally exponentially stable.

Remark 6. The Liapunov function of Theorem 1 need not be differentiable and not even be continuous. The regularity assumptions on $V(x)$ of Condition 1 of Theorem 2 are due to the proof of Theorem 2 itself.

4. Exponential stability and averaging

System (13) is equivalent to $\mathrm{d}x/\mathrm{d}\tau = \varepsilon f(x, \tau)$ by setting $\tau = \alpha t$ and $\varepsilon = 1/\alpha$. Let $x(\tau)$ be the solution of $\mathrm{d}x/\mathrm{d}\tau = \varepsilon f(x, \tau)$ with initial condition x_0 at τ_0. Let $y(\tau)$ be the solution of the averaged system $\mathrm{d}y/\mathrm{d}\tau = \varepsilon \bar{f}(y)$ with initial condition x_0 at τ_0 where $\forall y \in W$

$$\bar{f}(y) = \lim_{T \to \infty} \frac{1}{T} \int_{\tau_0}^{\tau_0 + T} f(y, \tau)\, \mathrm{d}\tau. \tag{31}$$

With appropriate smoothness and boundedness conditions on f, when $y(\tau) \in W$ $\forall \tau \in [\tau_0, \tau_0 + C/\varepsilon]$, it is shown (Sanders and Verhulst, 1985; Verhulst, 1990) that $\exists \varepsilon_0 > 0$ such that $\forall \varepsilon < \varepsilon_0$, $\|x(\tau) - y(\tau)\| = \mathrm{O}(\delta(\varepsilon))$ when $\tau \in [\tau_0, \tau_0 + C/\varepsilon]$ with

$$\delta(\varepsilon) = \sup_{x \in W} \sup_{0 \le \varepsilon\tau \le C} \varepsilon \left\| \int_{\tau_0}^{\tau_0 + \tau} f(x, s) - \bar{f}(x)\, \mathrm{d}s \right\|. \tag{32}$$

The approximation result that $\|x(\tau) - y(\tau)\| = \mathrm{O}(\delta(\varepsilon))$ when $\tau \in [\tau_0, \tau_0 + C/\varepsilon]$ is independent of the stability properties of the averaged system. When the averaged system is continuously differentiable and has a linear approximation which is exponentially stable, it is known (Sanders and Verhulst, 1985) that $\|x(\tau) - y(\tau)\| = \mathrm{O}(\delta(\varepsilon))$ $\forall \tau \ge \tau_0$ when x_0 is in the region of attraction of the averaged system.

In addition to its properties related to approximations of the solution $x(\tau)$ of $\mathrm{d}x/\mathrm{d}\tau = \varepsilon f(x, \tau)$, averaging is also useful in the study of exponential stability properties of the equilibrium point $x = 0$ (see Hale, 1980; Khalil, 1996; M'Closkey and Murray, 1993). Exponential stability of the averaged system implies exponential stability of the original time-varying system $\mathrm{d}x/\mathrm{d}\tau = \varepsilon f(x, \tau)$ when ε is sufficiently small. This result may be established by means of a state transformation as has been done in the almost periodic case by Hale (1980) and in the general case by Khalil (1996). The approach presented in this paper is totally different. The converse theorem of Liapunov for exponential stability guarantees the existence of a Liapunov function $V(x)$ whose derivative along the flow of the averaged system is negative definite. This $V(x)$ is used to establish, by Theorems 1 and 2, exponential stability of $\dot{x} = f(x, \alpha t)$ for α sufficiently large. In general, the derivative of $V(x)$ along the flow of $\dot{x} = f(x, \alpha t)$ may take positive and negative values which precludes the use of a classical theorem of Liapunov. But for α sufficiently large, we prove the existence of a $v > 0$, a $T > 0$ and a sequence of times t_k^*, satisfying the conditions of Theorem 1, such that $\forall k \in \mathbb{Z}$: $V(x(t_{k+1}^*)) - V(x(t_k^*)) \le -v\|x(t_k^*)\|^2$.

4.1. Averaging and local exponential stability

Theorem 3. *Consider the system*

$$\dot{x}(t) = f(x, \alpha t) \tag{33}$$

which satisfies the existence and uniqueness conditions mentioned in Section 2 with $f: W \times \mathbb{R} \to \mathbb{R}^n$, W open, $W \subset \mathbb{R}^n$, $0 \in W$ and $f(0, t) = 0$ $\forall t \in \mathbb{R}$. There exists $\alpha_1 > 0$ such that $\forall \alpha > \alpha_1$ system (33) *has an exponentially stable equilibrium point $x = 0$ if*

Condition 1: *The averaged system of $\dot{x}(t) = f(x, t)$, i.e., $\dot{x}(t) = \bar{f}(x)$ exists $\forall x \in W$ with*

$$\bar{f}(x) = \lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{+T} f(x, t)\, \mathrm{d}t. \tag{34}$$

The averaged system has an exponentially stable null solution, i.e., $\exists\ k > 1$, $\gamma > 0$ *and* $r_0 > 0$ *such that for all* $x_0 \in B_{r_o}(0)$

$$\|\Phi(t, t_0, x_0)\| \le k e^{-\gamma(t-t_0)}\|x_0\| \quad \forall t \ge t_0. \tag{35}$$

Here $\Phi(t, t_0, x_0)$ *denotes the solution of the averaged system at time t with initial condition* x_0 *at* t_0 *and* $\|x\|^2 = x^{\mathrm{T}}x$. *Here,* r_0 *is chosen such that the open sphere* $B_{kr_o}(0) \subset W$ *and Eq.* (35) *implies that* $\Phi(t, t_0, x_0) \in B_{kr_o}(0)$ *for every* $x_0 \in B_{r_o}(0)$, *for every* t_0 *and for all* $t \ge t_0$.

Condition 2: *The averaged system is twice continuously differentiable in* $B_{kr_o}(0)$ *and there exists a* $M_1 > 0$ *and a* $M_2 > 0$ *such that* $\forall x \in B_{kr_o}(0)\ \forall i, j, l \in \{1, \ldots, n\}$

$$\left|\frac{\partial \bar{f}_i}{\partial x_j}(x)\right| \le M_1 \quad \text{and} \quad \left|\frac{\partial^2 \bar{f}_i}{\partial x_j \partial x_l}(x)\right| \le M_2 \tag{36}$$

with $\bar{f} = (\bar{f}_1, \ldots, \bar{f}_n)^{\mathrm{T}}$.

Condition 3: *There exists a continuous nondecreasing function* $M(\cdot): [0, +\infty[\to [0, +\infty[$ *with* $\lim_{\sigma\to\infty} M(\sigma)/\sigma = 0$ *such that* $\forall x \in B_{r_o}(0)$, $\forall t_1, t_2 \in \mathbb{R}$ $(t_2 > t_1)$:

$$\left\|\int_{t_1}^{t_2} (f(x, t) - \bar{f}(x))\,\mathrm{d}t\right\| \le M(t_2 - t_1)\|x\|. \tag{37}$$

Proof. I: The proof relies upon Theorem 2. When the averaged system $\dot{x}(t) = \bar{f}(x)$ is exponentially stable, then the classical converse theorems of Liapunov (Khalil, 1996, pp. 148–152) guarantee the existence of an autonomous Liapunov function $V(x)$ satisfying the inequalities

$$c_1\|x\|^2 \le V(x) \le c_2\|x\|^2, \tag{38}$$

$$\frac{\partial V}{\partial x}(x)\bar{f}(x) \le -c_3\|x\|^2, \tag{39}$$

$$\left\|\frac{\partial V}{\partial x}(x)\right\| \le c_4\|x\|, \tag{40}$$

for all $x \in B_{r_o}(0)$ and for some positive constants c_1, c_2, c_3 and c_4. By taking $U = B_{r_o}(0)$, Condition 1 of Theorem 2 is satisfied by setting $\lambda_{\min} = c_1$, $\lambda_{\max} = c_2$ and proving the Lipschitz property of $\partial V/\partial x(x)$ on $B_{r_o}(0)$ since $\partial V/\partial x(0) = 0$ by Eq. (40).

II: In order to prove the Lipschitz property of $\partial V/\partial x(x)$ as required by Condition 1 of Theorem 2, we rely on (Khalil, 1996, Lemma 2.2, p. 71). When $\partial V/\partial x(x)$ and its Jacobian matrix exist and are continuous on $B_{r_o}(0)$ and there exists a constant L such that $\forall x \in B_{r_o}(0)$: $\|J(x)\| \le L$ where $J(x) = (J_{ij}(x))$ with $J_{ij}(x) = \partial^2 V/\partial x_i \partial x_j(x)$ then $\partial V/\partial x(x)$ is Lipschitz on $B_{r_o}(0)$ with Lipschitz constant L.

- First, we prove existence and continuity of $\partial V/\partial x(x)$ and its Jacobian. Since $\bar{f}(x)$ is twice continuously differentiable, by Hirsch and Smale (1974, Theorem 2, p. 302) $\Phi(\tau, 0, x)$ $(\forall \tau > 0\ \forall x \in B_{r_o}(0))$ is twice continuously differentiable with respect to x, i.e., $\partial\Phi_m/\partial x_i(\tau, 0, x)$ and $\partial^2\Phi_m/\partial x_i \partial x_j(\tau, 0, x)$ exist and are continuous $\forall i, j, m \in \{1, \ldots, n\}$. The function $V(x)$ in the converse theorem of Liapunov (Khalil, 1996) is given as (with an appropriate T_V)

$$\int_0^{T_V} \Phi(\tau, 0, x)^{\mathrm{T}}\Phi(\tau, 0, x)\,\mathrm{d}\tau = \sum_{m=1}^{n}\int_0^{T_V} \Phi_m^2(\tau, 0, x)\,\mathrm{d}\tau.$$

By taking the partial derivative of $V(x)$ with respect to x_i $\forall i \in \{1, \ldots, n\}$, one obtains that $\forall x \in B_{r_o}(0)$

$$\frac{\partial V}{\partial x_i}(x) = \sum_{m=1}^{n}\int_0^{T_V} 2\frac{\partial \Phi_m}{\partial x_i}(\tau, 0, x)\,\Phi_m(\tau, 0, x)\,\mathrm{d}\tau. \tag{41}$$

By taking the partial derivative of Eq. (41) with respect to x_j, one obtains that $\partial^2 V/\partial x_i \partial x_j(x)$ equals

$$2\sum_{m=1}^{n}\int_0^{T_V} \frac{\partial^2\Phi_m}{\partial x_i \partial x_j}(\tau, 0, x)\Phi_m(\tau, 0, x) + \frac{\partial \Phi_m}{\partial x_i}(\tau, 0, x)\,\frac{\partial \Phi_m}{\partial x_j}(\tau, 0, x)\,\mathrm{d}\tau. \tag{42}$$

The existence and continuity of $\Phi_m(\tau, 0, x)$, $\partial\Phi_m/\partial x_i(\tau, 0, x)$ and $\partial^2\Phi_m/\partial x_i\partial x_j(\tau, 0, x)$ imply by Eqs. (41) and (42) that $\partial V/\partial x_i(x)$ and $\partial^2 V/\partial x_i \partial x_j(x)$ *exist and are continuous in* x, $\forall x \in B_{r_o}(0)$.

- Second, we prove the boundedness of the Jacobian of $\partial V/\partial x(x)$, i.e., $J(x)$. We also calculate a suitable estimate of L because it will be useful to estimate the region of convergence of $\dot{x} = f(x, \alpha t)$ and α_1 as pointed out in Remark 10. Appendix A studies the flow of the averaged system. By Appendix A, one obtains that $\forall x \in B_{r_o}(0)$ and $\forall T', \tau > 0$ with $\tau \le T'$ that

$$\begin{aligned} &\sum_{i,j=1}^{n}\left|\frac{\partial \Phi_i}{\partial x_j}(\tau, 0, x)\right| \le M_3(T'), \\ &\sum_{i,j,l=1}^{n}\left|\frac{\partial^2 \Phi_i}{\partial x_j \partial x_l}(\tau, 0, x)\right| \le M_4(T'). \end{aligned} \tag{43}$$

Relying on Eqs. (42) and (43), Appendix B studies the Liapunov function $V(x)$. By Eq. (B.2) in Appendix B

$$\sum_{i,j=1}^{n}|J_{ij}(x)| = \sum_{i,j=1}^{n}\left|\frac{\partial^2 V}{\partial x_i \partial x_j}(x)\right| \le M_5(r_0) \tag{44}$$

which implies that the Jacobian matrix $J(x) = (J_{ij}(x))$ of $\partial V/\partial x(x)$ is bounded on $B_{r_o}(0)$, i.e., $\forall x \in B_{r_o}(0)$: $\|J(x)\| \le M_5(r_0)$.

- By (Khalil, 1996, Lemma 2.2, p. 71), $\partial V/\partial x(x)$ is Lipschitz on $B_{r_o}(0)$ with Lipschitz constant $L = M_5(r_0)$. This Lipschitz condition and Eq. (38) imply that Condition 1 of Theorem 2 is fulfilled with $U = B_{r_o}(0)$.

III: In order to verify Condition 2 of Theorem 2, take $t_0^{\diamond} \in \mathbb{R}$ and then $\forall T^f \in \mathbb{R}_0^+$

$$\int_{t_0^{\diamond}}^{t_0^{\diamond}+T^f} f(x, t)\,\mathrm{d}t = T^f\bar{f}(x) + \int_{t_0^{\diamond}}^{t_0^{\diamond}+T^f} (f(x, t) - \bar{f}(x))\,\mathrm{d}t.$$

By Eqs. (37) and (40), one obtains that $\forall x \in B_{r_o}(0)$, $\forall T^f \in \mathbb{R}_0^+$

$$\left|\frac{\partial V}{\partial x}(x)\int_{t_0^\diamond}^{t_0^\diamond + T^f}(f(x,t)-\bar{f}(x))\,\mathrm{d}t\right| \le \left\|\frac{\partial V}{\partial x}(x)\right\| M(T^f)\|x\| \le c_4 M(T^f)\|x\|^2.$$

This implies by Eq. (39) that

$$\frac{\partial V}{\partial x}(x)\int_{t_0^\diamond}^{t_0^\diamond + T^f} f(x,t)\,\mathrm{d}t \le (-c_3 T^f + c_4 M(T^f))\|x\|^2. \quad (45)$$

Since $M(T)/T$ is continuous and $\lim_{T\to\infty} M(T)/T = 0$, it is possible to take a T^f large enough such that $M(T^f)/T^f < c_3/c_4$ and $\forall T \ge T^f : M(T)/T < c_3/c_4$. Define $t_k^\diamond := t_0^\diamond + kT^f$ $\forall k \in \mathbb{Z}$. By repeating the argument leading to Eq. (45) one obtains that $\forall k \in \mathbb{Z}$

$$\frac{\partial V}{\partial x}(x)\int_{t_k^\diamond}^{t_{k+1}^\diamond} f(x,t)\,\mathrm{d}t \le (-c_3 T^f + c_4 M(T^f))\|x\|^2. \quad (46)$$

Therefore, Eq. (46) implies that Condition 2 of Theorem 2 is fulfilled with $\upsilon = c_3 T^f - c_4 M(T^f) > 0$, $T^\diamond = T^f$ and $U = B_{r_o}(0)$. By Theorem 2, $\exists \alpha_1 > 0$ such that the system $\dot{x}(t) = f(x, \alpha t)$ is exponentially stable if $\alpha > \alpha_1$. □

Remark 7. The regularity conditions imposed on $\bar{f}(x)$ by Condition 2 of Theorem 3 do not necessarily derive from the smoothness conditions of $f(x,t)$. Indeed, let $h: \mathbb{R} \to \mathbb{R}$ be a 1-periodic function of class C^∞, i.e., all the higher-order derivatives exist and are continuous. In the one-dimensional case $f(x,t) = h(xt)$ is of class C^∞ with respect to x and t. Since by Eq. (34),

$$\bar{f}(x) = \lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{+T} h(xt)\,\mathrm{d}t,$$

one obtains that $\bar{f}(0) = h(0)$ and $\forall x \ne 0$ that $\bar{f}(x) = \int_0^1 h(y)\,\mathrm{d}y$. Therefore, $\bar{f}(x)$ is discontinuous at 0 when $\int_0^1 h(y)\,\mathrm{d}y \ne h(0)$. This example illustrates that $\bar{f}(x)$ may be discontinuous even if $f(x,t)$ is of class C^∞ with respect to x and t.

Remark 8. In case $\dot{x} = f(x,t)$ is periodic in t with period T_f independent of x, the relationship between the smoothness conditions on $f(x,t)$ and the regularity conditions on $\bar{f}(x)$ is more transparant.

- When $f = (f_1, \ldots, f_n)^{\mathrm{T}}$ and its partial derivatives with respect to x up to the second order exist and there exists a $M_1 > 0$ and a $M_2 > 0$ such that $\forall x \in B_{kr_o}(0)$, $\forall t$ and $\forall i, j, l \in \{1, \ldots, n\}$:

$$\left|\frac{\partial f_i}{\partial x_j}(x,t)\right| \le M_1 \quad \text{and} \quad \left|\frac{\partial^2 f_i}{\partial x_j \partial x_l}(x,t)\right| \le M_2$$

then Eq. (36) is satisfied with the same bounds M_1 and M_2. Indeed, by Eq. (34) and the periodicity of $f(x,t)$, one obtains that $\forall i \in \{1, \ldots, n\}$:

$$\bar{f}_i(x) = \frac{1}{T_f}\int_0^{T_f} f_i(x,t)\,\mathrm{d}t.$$

By taking the partial derivative with respect to x_j of $\bar{f}_i(x)$ and taking the absolute value, one obtains that $\forall i, j \in \{1, \ldots, n\}$ $\forall x \in B_{kr_o}(0)$

$$\left|\frac{\partial \bar{f}_i}{\partial x_j}(x)\right| \le \frac{1}{T_f}\int_0^{T_f}\left|\frac{\partial f_i}{\partial x_j}(x,t)\right|\mathrm{d}t \le M_1. \quad (47)$$

By taking the second-order partial derivative with respect to x_j and x_l of $\bar{f}_i(x)$ and taking the absolute value, one obtains that $\forall i, j, l \in \{1, \ldots, n\}$ $\forall x \in B_{kr_o}(0)$: $|\partial^2 \bar{f}_i / \partial x_j \partial x_l(x)| \le M_2$.

- Assume $\forall x \in B_{kr_o}(0)$, $\forall \varepsilon > 0 : \exists \delta(\varepsilon, x) > 0$ such that $\|x - y\| < \delta(\varepsilon, x)$ with $y \in B_{kr_o}(0)$ implies that $\forall t$: $|f_i(x,t) - f_i(y,t)| < \varepsilon$,

$$\left|\frac{\partial f_i}{\partial x_j}(x,t) - \frac{\partial f_i}{\partial x_j}(y,t)\right| < \varepsilon$$

and

$$\left|\frac{\partial^2 f_i}{\partial x_j \partial x_l}(x,t) - \frac{\partial^2 f_i}{\partial x_j \partial x_l}(y,t)\right| < \varepsilon$$

$\forall i, j, l \in \{1, \ldots, n\}$

then $\bar{f}$ is twice continuously differentiable. Indeed,

$$|\bar{f}_i(x) - \bar{f}_i(y)| \le \frac{1}{T_f}\int_0^{T_f} |f_i(x,t) - f_i(y,t)|\,\mathrm{d}t < \varepsilon. \quad (48)$$

By repeating the same argument one obtains that $\|x - y\| < \delta(\varepsilon, x)$ with $x, y \in B_{kr_o}(0)$ implies that

$$\left|\frac{\partial \bar{f}_i}{\partial x_j}(x) - \frac{\partial \bar{f}_i}{\partial x_j}(y)\right| < \varepsilon$$

and

$$\left|\frac{\partial^2 \bar{f}_i}{\partial x_j \partial x_l}(x) - \frac{\partial^2 \bar{f}_i}{\partial x_j \partial x_l}(y)\right| < \varepsilon$$

$\forall i, j, l \in \{1, \ldots, n\}$,

i.e., $\bar{f}$ is twice continuously differentiable.

Remark 9. Notice that the conditions on $f(x,t)$ as stated in Remark 8, which recover the smoothness conditions required by (Khalil, 1996, p. 332), are sufficient conditions. They are not necessary to guarantee Condition 2 of Theorem 3. This is easily illustrated by means of an example. When $f_1: \mathbb{R} \to \mathbb{R}$ and $f_2: \mathbb{R} \to \mathbb{R}$ are bounded 1-periodic functions with $\int_0^1 f_1(\tau)\,\mathrm{d}\tau = -1$ and $\int_0^1 f_2(\tau)\,\mathrm{d}\tau = 0$, consider the system $\dot{x} = f(x,t) = f_1(t)\,x + f_2(t)\,F(x)$ where $F(x)$ is Lipschitz continuous but not differentiable. The averaged system $\dot{x} = \bar{f}(x) = -x$ satisfies Condition 2 of Theorem 3 although $F(x)$ and therefore also $f(x,t)$ are not differentiable in x.

Remark 10. Theorem 3 not only proves that exponential stability of the averaged system implies exponential stability of the original fast time-varying system $\dot{x} = f(x, \alpha t)$ for $\alpha > \alpha_1$, the theorem also gives an estimate of the region of exponential stability and an estimate of α_1.

Indeed, the proof of Theorem 3 with $\bar{B}_{r_o}(0) \subset B_{kr_o}(0) \subset W$ relies on Theorem 2 with $U = B_{r_o}(0)$. Because of Theorem 2 and Remark 5 with $T^{\diamond} = T^f$, $\lambda_{\min} = c_1$ and $\lambda_{\max} = c_2$, it is clear that $B_{\delta''}(0)$ belongs to the region of exponential stability of Eq. (33) when $\delta'' < e^{-3KT^f/\alpha}\sqrt{c_1/c_2}\, r_0$ with T^f large enough such that $c_3 T^f - c_4 M(T^f) > 0$.

Here, K is an upper bound for the Lipschitz functions $l_x(t)$ of $f(x, t)$ on $\bar{B}_{r_o}(0)$. By Remark 4, α_1 is given by Eq. (30) with K, $K_v = M_5(r_0)$, $\upsilon = c_3 T^f - c_4 M(T^f)$ and $T^{\diamond} = T^f$.

4.2. Averaging and semiglobal exponential stability

Even when Eq. (33) is globally Lipschitz and the conditions of Theorem 3 hold globally, Theorem 3 provides *no* $\alpha_1 > 0$ such that $\forall \alpha > \alpha_1 : \dot{x} = f(x, \alpha t)$ is globally exponentially stable. Indeed, Theorem 3 relies on Theorem 2. By Theorem 2 and Remark 5, a global exponential stability result for $\dot{x} = f(x, \alpha t)$ does not only require that the conditions of Theorem 2 hold globally, also a global Lipschitz property on $\partial V/\partial x(x)$ is needed. This global Lipschitz condition on $\partial V/\partial x(x)$ is not satisfied since by Eq. (44), $K_v = M_5(r_0)$ grows unbounded when r_0 goes to infinity. It is however possible to prove a semiglobal exponential stability result:

Theorem 4. *When the conditions of Theorem* 3 *hold globally then* $\dot{x} = f(x, \alpha t)$ *is semiglobally exponentially stable, i.e.,* $\forall R > 0$, $\exists \alpha_2(R)$ *such that* $\forall \alpha > \alpha_2(R) : \dot{x}(t) = f(x, \alpha t)$ *is exponentially stable and* $B_R(0)$ *belongs to the region of exponential stability.*

Proof. Since the conditions on $\dot{x} = \bar{f}(x)$ hold globally, Eqs. (38)–(40) hold globally for some positive constants c_1, c_2, c_3 and c_4. In order to prove the semiglobal exponential stability result, take for every $R > 0$, $r_0 = 2R\sqrt{c_2/c_1}$, and apply Theorem 3 and Remark 10 with this specific choice of r_0. Because of Theorem 3 and Remark 10, $\dot{x} = f(x, \alpha t)$ is exponentially stable when $\alpha > \alpha_1(R)$ with $\alpha_1(R)$ satisfying Eq. (30) with $K(r_0)$, $K_v = M_5(r_0)$, $\upsilon = c_3 T^f - c_4 M(T^f)$ and $T^{\diamond} = T^f$. Here $K(r_0)$, the notation emphasizes its dependence on r_0, denotes the Lipschitz constant of $f(x, t)$ on $\bar{B}_{r_o}(0)$.

An open ball with center 0 and radius δ'' belongs to the region of exponential stability when $\delta'' < e^{-3K(r_o)T^f/\alpha}2R$. When $\alpha > \alpha_2(R) = \max\{\alpha_1(R), 3K(r_0)T^f/\ln 2\}$ then $\dot{x} = f(x, \alpha t)$ is exponentially stable and by Remark 10, $B_R(0)$ belongs to the region of exponential stability since $R < e^{-3K(r_o)T^f/\alpha}2R$. This establishes the semiglobal exponential stability property of $\dot{x} = f(x, \alpha t)$. □

Remark 11. Along the lines of the proof of Theorem 4 no global exponential stability of $\dot{x} = f(x, \alpha t)$ has been established. Indeed, as pointed out in the proof of Theorem 4, for every $R > 0$, $\exists$ finite $\alpha_1(R)$ which satisfies Eq. (30). But when R increases, $K(r_0)$ is nondecreasing, $M_5(r_0) = M_5(2R\sqrt{c_2/c_1})$ and $\alpha_1(R)$ increase and $\lim_{R\to+\infty} \alpha_1(R) = +\infty$ such that also $\lim_{R\to+\infty}\alpha_2(R) = +\infty$.

5. Conclusions

The main theorem of this paper gives a sufficient condition for exponential stability of a differential equation. This result is useful for the study of exponential stability of fast time-varying systems: exponential stability of $\dot{x} = \bar{f}(x)$ implies that $\exists \alpha_1 > 0$ such that $\forall \alpha > \alpha_1$: $\dot{x} = f(x, \alpha t)$ is exponentially stable. Comparing these results with the results available in the literature, it becomes clear that

- The averaging results of the present paper are concerned with exponential stability of a time-varying system. The averaging results established by Hapaev (1993) are related to instability, stability or asymptotic stability properties of the original system, not to exponential stability properties.
- The averaging results established by Hale (1980) are restricted to periodic and almost periodic systems $\dot{x} = \varepsilon f(x, t, \varepsilon)$. Hale (1980, p. 194, Theorems 3.1 and 3.2) proves that for ε sufficiently small, there is a unique periodic or almost periodic solution of the original system, in a neighborhood of the equilibrium of the averaged system. This unique solution is uniformly asymptotically stable when the linearization of the averaged system is exponentially stable. In our framework, this unique solution is the zero solution since $f(0, t) = 0\ \forall t$. Notice however that in the present paper no periodicity neither almost periodicity is required.
- The results in the present paper are related to the averaging results established by Khalil (1996). There are differences in the regularity assumptions, i.e., the smoothness conditions on Theorem 3 are mainly focused on the averaged system $\bar{f}(x)$ whereas Khalil (1996) formulates smoothness conditions on $f(x, t)$ itself. There are also some technical differences in the convergence condition, but the main point is that our proof technique is cast in a Liapunov context. The use of Theorem 1 with a Liapunov function $V(x)$ which decreases stepwise makes it not necessary to use a state transformation as typically done in averaging theory, (see Hale, 1980; Khalil, 1996; Sanders and Verhulst, 1985; Verhulst, 1990).

- The results obtained in the present paper are different from the results obtained by M'Closkey and Murray (1993) (see also M'Closkey, 1997). M'Closkey and Murray prove that asymptotic stability of the averaged system of a periodic zero-order homogeneous system implies its exponential stability with respect to the homogeneous norm. The systems discussed in the present paper are not necessarily homogeneous and we obtain exponential stability with respect to a classical norm.

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Appendix A

The solution $\Phi(\tau, 0, x)$ of the averaged system $\dot{x} = \bar{f}(x)$ satisfies $\forall x \in B_{r_o}(0)$, $\forall \tau > 0$ and $\forall i \in \{1, \dots, n\}$

$$\Phi_i(\tau, 0, x) = x_i + \int_0^\tau \bar{f}_i(\Phi_1(s, 0, x), \dots, \Phi_n(s, 0, x))\,\mathrm{d}s$$

where $\Phi = (\Phi_1, \dots, \Phi_n)^{\mathrm{T}}$. Taking the partial derivative with respect to x_j, one obtains that $\partial\Phi_i/\partial x_j(\tau, 0, x)$ equals

$$\delta_{ij} + \int_0^\tau \sum_{m=1}^n \frac{\partial \bar{f}_i}{\partial \Phi_m}(\Phi(s, 0, x)) \frac{\partial \Phi_m}{\partial x_j}(s, 0, x)\,\mathrm{d}s. \tag{A.1}$$

Taking the absolute value of equation (A.1), invoking the triangle inequality and taking the sum of expression (A.1) over all $i, j \in \{1, \dots, n\}$, one obtains by Eq. (36) with $\Phi(s, 0, x) \in B_{kr_o}(0)$ that

$$\sum_{i,j=1}^n \left|\frac{\partial \Phi_i}{\partial x_j}(\tau, 0, x)\right|$$

is less than or equal to

$$\sum_{i,j=1}^n \delta_{ij} + M_1 \sum_{i,j=1}^n \int_0^\tau \sum_{m=1}^n \left|\frac{\partial \Phi_m}{\partial x_j}(s, 0, x)\right| \mathrm{d}s. \tag{A.2}$$

This implies that

$$\sum_{i,j=1}^n \left|\frac{\partial \Phi_i}{\partial x_j}(\tau, 0, x)\right| \le n + nM_1 \int_0^\tau \sum_{m,j=1}^n \left|\frac{\partial \Phi_m}{\partial x_j}(s, 0, x)\right| \mathrm{d}s.$$

Application of the Bellman–Gronwall lemma gives $\forall T', \tau \ge 0$ with $\tau \le T'$ that

$$\sum_{i,j=1}^n \left|\frac{\partial \Phi_i}{\partial x_j}(\tau, 0, x)\right|$$

is less than or equal to $n\mathrm{e}^{nM_1T'} =: M_3(T')$.

The partial derivative of the expression (A.1) with respect to x_l equals

$$\sum_{m=1}^n \int_0^\tau \left(\sum_{p=1}^n \frac{\partial^2 \bar{f}_i}{\partial \Phi_m \partial \Phi_p}(\Phi(s, 0, x)) \frac{\partial \Phi_p}{\partial x_l}(s, 0, x)\right) \frac{\partial \Phi_m}{\partial x_j}(s, 0, x) + \frac{\partial \bar{f}_i}{\partial \Phi_m}(\Phi(s, 0, x)) \frac{\partial^2 \Phi_m}{\partial x_j \partial x_l}(s, 0, x)\,\mathrm{d}s.$$

Taking the absolute value, invoking the triangle inequality and by Eq. (36), one obtains with $\Phi(s, 0, x) \in B_{kr_o}(0)$ that $\forall i, j, l \in \{1, \dots, n\}$, $\forall T', \tau \ge 0$ with $\tau \le T'$ that

$$\left|\frac{\partial^2 \Phi_i}{\partial x_j \partial x_l}(\tau, 0, x)\right|$$

is less than or equal to

$$\sum_{m=1}^n \int_0^\tau M_2 M_3^2(T') + M_1 \left|\frac{\partial^2 \Phi_m}{\partial x_j \partial x_l}(s, 0, x)\right| \mathrm{d}s. \tag{A.3}$$

Taking the sum of the expression (A.3) over all $i, j, l \in \{1, \dots, n\}$ proves that

$$\sum_{i,j,l=1}^n \left|\frac{\partial^2 \Phi_i}{\partial x_j \partial x_l}(\tau, 0, x)\right|$$

is less than or equal to

$$\tau n^4 M_2 M_3^2(T') + nM_1 \sum_{m,j,l=1}^n \int_0^\tau \left|\frac{\partial^2 \Phi_m}{\partial x_j \partial x_l}(s, 0, x)\right| \mathrm{d}s.$$

Application of the Bellman–Gronwall lemma gives $\forall T', \tau \ge 0$ with $\tau \le T'$ that

$$\sum_{i,j,l=1}^n \left|\frac{\partial^2 \Phi_i}{\partial x_j \partial x_l}(\tau, 0, x)\right|$$

is less than or equal to $T' n^4 M_2 M_3^2(T')\,\mathrm{e}^{nM_1T'} =: M_4(T')$.

Appendix B

By adding the absolute values of Eq. (42) $\forall i, j \in \{1, \dots, n\}$ with $\Phi(\tau, 0, x) \in B_{kr_o}(0)$, one obtains that

$$\sum_{i,j=1}^n \left|\frac{\partial^2 V}{\partial x_i \partial x_j}(x)\right|$$

is less than or equal to

$$2\int_0^{T_V} \sum_{i,j,m=1}^n \left|\frac{\partial^2 \Phi_m}{\partial x_i \partial x_j}(\tau, 0, x)\right| kr_0 + \sum_{m=1}^n \left(\left(\sum_{i=1}^n \left|\frac{\partial \Phi_m}{\partial x_i}(\tau, 0, x)\right|\right)\left(\sum_{j=1}^n \left|\frac{\partial \Phi_m}{\partial x_j}(\tau, 0, x)\right|\right)\right) \mathrm{d}\tau. \tag{B.1}$$

By Eq. (43),

$$\sum_{j=1}^n \left|\frac{\partial \Phi_m}{\partial x_j}(\tau, 0, x)\right| \le M_3(T_\mathrm{V}) \quad \forall m \in \{1, \dots, n\}$$

such that equation (B.1) implies that

$$\sum_{i,j=1}^{n} \left| \frac{\partial^2 V}{\partial x_i \partial x_j}(x) \right|$$

is less than or equal to

$$2T_V M_4(T_V) k r_0 + 2\int_0^{T_V} \left(\sum_{m,i=1}^{n} \left| \frac{\partial \Phi_m}{\partial x_i}(\tau, 0, x) \right| \right) M_3(T_V)\, d\tau$$

$$\leq 2T_V (M_4(T_V) k r_0 + M_3^2(T_V)) =: M_5(r_0). \qquad \text{(B.2)}$$

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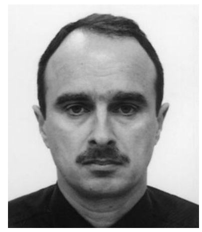

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