A New Asymptotic Stability Criterion for Nonlinear Time-Variant Differential Equations

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***Abstract*—A new sufficient condition for asymptotic stability of ordinary differential equations is proposed. Unlike classical Liapunov theory, the time derivative along solutions of the Liapunov function may have positive and negative values. The classical Liapunov approach may be regarded as an infinitesimal version of the present theorem. Verification in practical problems is harder than in the classical case; an example is included in order to indicate how the present theorem may be applied.**

***Index Terms*—Liapunov, nonlinear dynamics, time-varying systems.**

I. Introduction

The classical Liapunov approach to asymptotic stability of an equilibrium point of a dynamical system $\dot{x}(t) = f(x,t)$ requires the existence of a positive definite Liapunov function $V(x,t)$ with $\dot{V}(x,t)$ negative definite. When $\dot{V}(x,t) \leq 0$, stability rather than asymptotic stability follows. In this case, the Barbashin–Krasovskii theorem or LaSalle's invariance principle may be helpful in proving asymptotic stability; indeed, there is convergence to the largest invariant set in $E = \{x : \dot{V}(x) = 0\}$ when the differential equation is autonomous, and extension to the periodic case is possible. When the system is not periodic, Narendra and Annaswamy [4] show that with $\dot{V}(x,t) \leq 0$ uniform asymptotic stability can be proven if there exists a T such that $\forall t : \int_t^{t+T} \dot{V}(x(\tau),\tau)\, d\tau \leq -\gamma(\|x(t)\|) < 0$ where $\gamma(\cdot)$ is a positive monotonic function on $\mathbb{R}^+$ which is zero at the origin. Weaker conditions than the Narendra–Annaswamy conditions still implying asymptotic stability have recently been obtained [1], [2]. In these results one has examined the role of T and its dependence on the initial conditions (x_0, t_0). This paper is to a great deal inspired by the result of Narendra and Annaswamy [4]. Basically, we claim that *the negative semidefiniteness condition on $\dot{V}(x,t)$ is no longer needed*. Roughly stated: the origin is asymptotically stable under the condition that for a positive definite decrescent $V(x,t)$, $\exists T > 0, \forall t_0, \forall x_0 : V(x(t_0+T), t_0+T) - V(x_0,t_0) \leq -\gamma(\|x_0\|) < 0$; $x(t_0+T)$ is the solution of $\dot{x}(t) = f(x,t)$ evaluated at $t_0 + T$ with initial condition x_0 at t_0.

II. The Main Theorem

Consider

$$\dot{x}(t) = f(x,t) \tag{1}$$

with $f : W \times \mathbb{R} \to \mathbb{R}^n$, W open, $W \subset \mathbb{R}^n$. Let $f(0,t) = 0\ \forall t \in \mathbb{R}$ and $0 \in W$. Furthermore, we assume that conditions are imposed on (1) such that existence and uniqueness of its solutions are secured. Of these conditions, we single out the local Lipschitz condition because it will be used in the course of the proof of Lemma 1 and Theorem 1: f is locally Lipschitz, i.e., for $\forall x \in W$, $\exists$ a neighborhood $\mathcal{O}(x) \subset W$, such that the restriction $f|_{\mathcal{O}(x)\times\mathbb{R}} \neq f|_{\mathcal{O}(x)}$ is Lipschitz with Lipschitz function $l_x(t)$. We assume that $l_x(t)$ is bounded $\forall t \in \mathbb{R}$. We are now ready to state a lemma and the main theorem.

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Lemma 1: Let $U \subset W$ be an open neighborhood of the origin. Consider a closed ball $\bar{B}_\mu(0) \subset U$, then $\forall T > 0$, T finite: $\exists \mu' > 0$ such that $(x_0,t_0) \in B_{\mu'}(0) \times \mathbb{R}$ implies that $x(t) \in B_\mu(0)$ for $t \in [t_0, t_0+T]$. Here $x(t)$ is the solution of (1) evaluated at t with initial condition x_0 at t_0.

Proof: Consider a closed ball $\bar{B}_\mu(0)$ centered at the origin and with radius μ small enough such that $\bar{B}_\mu\ (0) \subset U$. Since $\bar{B}_\mu(0)$ is compact, it can be covered with a finite number of open convex neighborhoods $\mathcal{O}(x)$, $x \in \bar{B}_\mu(0)$. Let K be the maximum of the bounded Lipschitz functions $l_x(t)$ involved. Define $\mu' := \mu e^{-KT}$. Let $\bar{B}_{\mu'}(0)$ be the closed ball centered at the origin and with radius μ'. Notice that $\bar{B}_\mu(0)$ is convex and therefore $\forall x, y \in \bar{B}_\mu(0)$, the line segment joining x and y belongs to $\bar{B}_\mu(0)$. Denote this line segment as l_{xy}. Pick a neighborhood from the finite covering which contains x and denote it as $\mathcal{O}(x)$. When $y \in \mathcal{O}(x)$, one obtains immediately that $\|f(x,t) - f(y,t)\| \leq K\|x-y\|$. When $y \notin \mathcal{O}(x)$, define the point b_1 such that $\{b_1\} = b(\mathcal{O}(x)) \cap l_{xy}$ where $b(\mathcal{O}(x))$ is the boundary of $\mathcal{O}(x)$. Pick a neighborhood from the finite covering which contains b_1 and denote it as $\mathcal{O}(b_1)$. Pick a point $p_1 \in l_{xy} \cap \mathcal{O}(x) \cap \mathcal{O}(b_1)$, then $\|f(x,t) - f(p_1,t)\| \leq K\|x - p_1\|$. When $y \in \mathcal{O}(b_1)$, then $\|f(p_1,t) - f(y,t)\| \leq K\|p_1 - y\|$. When $y \notin \mathcal{O}(b_1)$, define b_2 such that $\{b_2\} = b(\mathcal{O}(b_1)) \cap l_{b_1 y}$. Pick a neighborhood from the finite covering which contains b_2 and denote it as $\mathcal{O}(b_2)$. Pick a point $p_2 \in l_{b_1 y} \cap \mathcal{O}(b_1) \cap \mathcal{O}(b_2)$, then $\|f(p_1,t) - f(p_2,t)\| \leq K\|p_1 - p_2\|$. This argument can be repeated, and since we start with a finite covering there exists a finite integer n_{xy} and related points b_i and p_i with $i \in \{1, \cdots, n_{xy}\}$ such that $y \in \mathcal{O}(b_{n_{xy}})$. One obtains that

$$\begin{gathered} \|f(x,t) - f(p_1,t)\| \leq K\|x - p_1\| \\ \vdots \\ \|f(p_i,t) - f(p_{i+1},t)\| \leq K\|p_i - p_{i+1}\| \\ \vdots \\ \|f(p_{n_{xy}},t) - f(y,t)\| \leq K\|p_{n_{xy}} - y\|. \end{gathered}$$

Since all p_i belong to the line segment l_{xy}, adding these inequalities and invoking the triangle inequality on the left-hand side implies that $\forall x, y \in \bar{B}_\mu(0)$ and $\forall t \in \mathbb{R}$

$$\|f(x,t) - f(y,t)\| \leq K\|x - y\|. \tag{2}$$

If $x(t)$ is the solution of (1) evaluated at t with initial condition $x_0 \in B_{\mu'}(0)$ at t_0 $(x(t_0) = x_0)$

$$x(t) = x(t_0) + \int_{t_0}^{t} f(x(s),s)\, ds, \qquad \forall t \geq t_0. \tag{3}$$

By the Lipschitz condition

$$\|x(t)\| \leq \|x(t_0)\| + \int_{t_0}^{t} K\|x(s)\|\, ds \tag{4}$$

and by the Bellman–Gronwall lemma

$$\|x(t)\| \leq \|x(t_0)\| e^{K(t-t_0)} \tag{5}$$

with $t \geq t_0$ and with the additional restriction that $x(t) \in \bar{B}_\mu(0)$ and thus that the Lipschitz condition may be applied. Consider the trajectory $x(t)$, $t \geq t_0$, with $x(t_0) \in B_{\mu'}(0)$. We will show that $x(t)$ satisfies (5), for $t \in [t_0, t_0+T]$. This is obviously satisfied if $x(t) \in B_\mu(0)$, $\forall t \geq t_0$. If $x(t) \notin B_\mu(0)$ for some $t > t_0$, there is

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by continuity of the flow a smallest $t_1 > t_0$ such that $x(t_1)$ belongs to the sphere $S_\mu(0)$. By application of the Lipschitz condition

$$\|x(t_1)\| \le \|x(t_0)\| + \int_{t_0}^{t_1} K\|x(s)\|\, ds \tag{6}$$

and by the Bellman–Gronwall lemma

$$\|x(t_1)\| \le \|x(t_0)\| e^{K(t_1-t_0)}. \tag{7}$$

With $\|x(t_1)\| = \mu$ and $\|x(t_0)\| < \mu'$ we immediately obtain that $t_1 - t_0 > T$, or $t_1 > t_0 + T$. Therefore, we conclude that (5) is true for $t \in [t_0, t_0 + T]$. This implies that if $x(t_0) \in B_{\mu'}(0) \subset U$ with $\mu' := \mu e^{-KT}$, then $x(t) \in B_\mu(0) \subset U$ for $t \in [t_0, t_0 + T]$. □

Remark 1: The proof of the lemma shows that $\mu' = \mu e^{-KT}$ is an appropriate choice of μ'. K is the maximum of the upper bounds of the Lipschitz functions $l_x(t)$ corresponding to a finite subcovering of $\bar{B}_\mu(0)$ which consists of neighborhoods $\mathcal{O}(x)$.

Theorem 1: Consider a function $V : U \times \mathbb{R} \to \mathbb{R}$ with $U \subset W$ an open neighborhood of the origin. We assume that the following additional conditions are satisfied.

- *Condition 1:* $V(x,t)$ is positive definite and decrescent; i.e., $V(0,t) = 0\ \ \forall t$ and $\forall x \in U : \alpha(\|x\|) \le V(x,t) \le \beta(\|x\|)$. The functions $\alpha(\cdot) : \mathbb{R}^+ \to \mathbb{R}$ and $\beta(\cdot) : \mathbb{R}^+ \to \mathbb{R}$ belong to class $\mathcal{K}$, i.e., they are strictly increasing continuous functions passing through the origin.
- *Condition 2:* $\exists$ finite $T > 0$ and a $\gamma(\cdot) : \mathbb{R}^+ \to \mathbb{R}$ which belongs to class $\mathcal{K}$, i.e., $\gamma(\cdot)$ is continuous, strictly increasing, and passing through the origin and an open set $U' \subset U$ which contains the origin such that $\forall (x,t) \in U'\backslash\{0\} \times \mathbb{R}$:

$$V(x(t+T), t+T) - V(x,t) \le -\gamma(\|x\|) < 0. \tag{8}$$

Then the equilibrium point $x = 0$ of (1) is uniformly asymptotically stable.

Proof—Uniform Stability: Consider a closed ball $\bar{B}_\epsilon(0)$ centered the origin and with radius ϵ small enough such that $\bar{B}_\epsilon(0) \subset U'$. Since $\bar{B}_\epsilon(0)$ is compact, it can be covered with a finite number of open neighborhoods $\mathcal{O}(x), x \in \bar{B}_\epsilon(0)$. Let K be the maximum of the bounded Lipschitz functions $l_x(t)$ involved. Define $\epsilon' := \epsilon e^{-KT}$. Let $\bar{B}_{\epsilon'}(0)$ be the closed ball centered at the origin and with radius ϵ'. Define $\delta' := \beta^{-1}(\alpha(\epsilon'))$ and consider the open ball $B_{\delta'}(0)$. For all $(x_0, t_0) \in B_{\delta'}(0) \times \mathbb{R}$, $x(t_0) = x_0$, $\beta(\|x(t_0)\|) < \alpha(\epsilon')$ and thus $V(x(t_0), t_0) < \alpha(\epsilon')$ by Condition 1. By Condition 2 one obtains $V((x(t_0 + T), t_0 + T) < \alpha(\epsilon')$. Since $\alpha(\|x(t_0+T)\|) \le V((x(t_0+T), t_0+T)$, by Condition 1, one obtains $\alpha(\|x(t_0 + T)\|) < \alpha(\epsilon')$ which implies that $\|x(t_0 + T)\| < \epsilon'$, and therefore $x(t_0 + T) \in U'$. By Condition 2 one obtains that $V((x(t_0 + 2T), t_0 + 2T) \le V((x(t_0 + T), t_0 + T) < \alpha(\epsilon')$. Since $\alpha(\|x(t_0 + 2T)\|) \le V((x(t_0 + 2T), t_0 + 2T)$, one knows that $\|x(t_0 + 2T)\| < \epsilon'$, and therefore $x(t_0 + 2T) \in U'$. By repeating this argument one obtains that $\forall n \in \mathbb{N}$: $V((x(t_0+nT), t_0+nT) < \alpha(\epsilon')$ implying that $\|x(t_0 + nT)\| < \epsilon'$ and so $x(t_0 + nT) \in U'$. Summarizing, we have shown that if $x(t_0) \in B_{\delta'}(0)$, then $\|x(t_0 + nT)\| < \epsilon'$, $\forall n \in \mathbb{N}$.

Notice that $\forall t \ge t_0$, $\exists n \in \mathbb{N}$ such that $t - t_0 - nT < T$. Since $\|x(t_0 + nT)\| < \epsilon' = \epsilon e^{-KT}$, $\forall n \in \mathbb{N}$, Lemma 1 and Remark 1 imply that $x(t) \in B_\epsilon(0)\ \forall t \ge t_0$ or equivalently $\|x(t)\| < \epsilon\ \forall t \ge t_0$.

In order to prove the uniform stability of the system, i.e.,

$$\forall \nu > 0, \qquad \exists \delta(\nu) > 0 : \|x(t_0)\| < \delta(\nu) \Rightarrow \|x(t)\| < \nu \tag{9}$$

$\forall t, t_0$ with $t \ge t_0$. Take $\epsilon > 0$ such that $\bar{B}_\epsilon \subset U' \cap \bar{B}_\nu(0)$. We simply take $\delta(\nu) = \delta' = \beta^{-1}(\alpha(\epsilon e^{-KT}))$. Then with $\|x(t_0)\| < \delta'$ one has $\|x(t_0 + nT)\| < \epsilon'\ \forall n \in \mathbb{N}$ and therefore $\|x(t)\| < \epsilon' e^{KT} = \epsilon \le \nu$ $\forall t \ge t_0$. □

Proof—Uniform Convergence: We need to show that there exists $\epsilon_1 > 0$ such that $\forall \epsilon_2 > 0$ there exists a $T(\epsilon_2) \ge 0$ with the property that if $\|x(t_0)\| < \epsilon_1$ (t_0 arbitrary), then $\|x(t)\| < \epsilon_2$ for all $t \ge t_0 + T(\epsilon_2)$. First we will prove that there exists $\epsilon_1 > 0$ such that $\forall \epsilon_2 > 0$ there exists a finite $k(\epsilon_2) \ge 0$ ($k(\epsilon_2) \in \mathbb{N}$) such that $\|x(t_0)\| < \epsilon_1$ implies $\|x(t_0 + k(\epsilon_2)T)\| < \delta_2'$, with $\delta_2' = \beta^{-1}(\alpha(\epsilon_2'))$ and $\epsilon_2' = \epsilon_2 e^{-KT}$.

Let $0 < \epsilon_1 < \beta^{-1}(\alpha(\epsilon e^{-KT}))$ with $\bar{B}_\epsilon(0) \subset U'$ and take an arbitrary $x(t_0) = x_0$ with $x(t_0) \in B_{\epsilon_1}(0)$. The uniform stability guarantees that $x(t) \in B_\epsilon(0) \subset U'\ \forall t \ge t_0$. Therefore, Condition 2 can be applied.

Since $\|x(t_0)\| < \epsilon_1$, $V(x(t_0), t_0) < \beta(\epsilon_1)$. By Condition 2, we know that $V(x(t_0 + (k+1)T), t_0 + (k+1)T) - V(x(t_0 + kT), t_0 + kT) \le -\gamma(\|x(t_0 + kT)\|) < 0$ with $x(t_0 + kT) \in U'\backslash\{0\}$, $\forall k \in \mathbb{N}$. Therefore $V(x(t_0 + kT), t_0 + kT)$ is decreasing by a value at least equal to $\gamma(\delta_2') > 0$ for each time-interval T as long as $\|x(t_0 + kT\|) \ge \delta_2'$.

Suppose that $\|x(t_0 + kT)\| \ge \delta_2'\ \forall k \in \mathbb{N}$, then for all $k \in \mathbb{N}$ $V(x(t_0 + (k+1)T), t_0 + (k+1)T) - V(x(t_0 + kT), t_0 + kT) \le -\gamma(\|x(t_0 + kT)\|) \le -\gamma(\delta_2')$. This implies that $V(x(t_0 + (k+1)T), t_0 + (k+1)T) - V(x(t_0), t_0) \le -(k+1)\gamma(\delta_2')$. Since the Liapunov function is positive-definite, $0 \le V(x(t_0 + (k+1)T), t_0 + (k+1)T) \le V(x(t_0), t_0) - (k+1)\gamma(\delta_2') \le \beta(\epsilon_1) - (k+1)\gamma(\delta_2')$, $\forall x(t_0) \in B_{\epsilon_1}(0)$ and $\forall k \in \mathbb{N}$. Since $\beta(\epsilon_1) > 0$ and $\gamma(\delta_2') > 0$ are finite, $\beta(\epsilon_1) - (k+1)\gamma(\delta_2')$ cannot be positive for every $k \in \mathbb{N}$. As a consequence, the assumption that $\|x(t_0 + kT)\| \ge \delta_2'\ \forall k \in \mathbb{N}$ is wrong and therefore $\exists k^* \in \mathbb{N}$ such that $\|x(t_0 + k^*T)\| < \delta_2'$. For this $k^* := k(\epsilon_2)$ (notice that k^* depends on δ_2' and therefore on ϵ_2), one obtains $\|x(t_0 + k^*T)\| < \delta_2'$.

Since $V(x(t_0 + k^*T), t_0 + k^*T) < \beta(\|x(t_0 + k^*T)\|) < \beta(\delta_2')$ and $\beta(\delta_2') = \alpha(\epsilon_2')$ and because of Condition 2, we obtain $\forall m \in \mathbb{N} \ge k^*$ that $V(x(t_0 + mT), t_0 + mT) < \alpha(\epsilon_2')$. Then, $\alpha(\|x(t_0 + mT)\|) < \alpha(\epsilon_2')$ and also $\|x(t_0 + mT)\| < \epsilon_2' = \epsilon_2 e^{-KT}$.

By Lemma 1 and Remark 1, we conclude that for all $t > t_0 + k^*T$, $\|x(t)\| < \epsilon_2$. With $T(\epsilon_2) := k^*T$ we have established uniform convergence to the origin and therefore also its uniform asymptotic stability. □

Remark 2: Lemma 1 and Remark 1 guarantee that there exists an open set U' (mentioned in Condition 2) such that $V(x(t+T), t+T)$ is defined if $(x,t) \in U'\backslash\{0\} \times \mathbb{R}$. This U', which contains the origin, can, e.g., be an open subset of $B_{\mu'}(0)$.

Remark 3: $\dot{V}(x,t)$ need not be differentiable [as in classical approaches in order to compute $\dot{V}(x,t)$] and not even be continuous. Conditions 1 and 2, where the functions $\alpha(\cdot)$, $\beta(\cdot)$, and $\gamma(\cdot)$, are continuous, provide sufficient information to prove uniform asymptotic stability.

Remark 4: In the case $V(x,t)$ is differentiable, the classical Liapunov theorem for asymptotic stability may be considered as an infinitesimal version of the theorem above obtained for T tending to zero.

Remark 5: If (1) is defined on $\mathbb{R}^n \times \mathbb{R}$ and if the Liapunov function $V(x,t)$ is defined on $\mathbb{R}^n \times \mathbb{R}$ radially unbounded and satisfies Conditions 1 and 2 of the theorem on $\mathbb{R}^n$, then one obtains global asymptotic stability of the origin. Global uniform asymptotic stability requires the existence of a $\mathcal{K}_\infty$ function $\delta(\cdot)$ ([3, p. 135]) such that $\forall t, t_0$ with $t \ge t_0$

$$\forall \epsilon > 0 : \|x(t_0)\| < \delta(\epsilon) \Rightarrow \|x(t)\| < \epsilon \tag{10}$$

and

$$\forall \epsilon_1, \epsilon_2 > 0 : \exists T(\epsilon_1, \epsilon_2) > 0 : \|x(t_0)\| < \epsilon_1 \Rightarrow \|x(t)\| < \epsilon_2 \tag{11}$$

$\forall t_0$, $\forall t \ge t_0 + T(\epsilon_1, \epsilon_2)$. Notice that (10) is not readily implied by Conditions 1 and 2 of the theorem on $\mathbb{R}^n$. However, if we introduce a global Lipschitz condition, i.e., $\exists K > 0$ such that $\forall x, y \in \mathbb{R}^n$, $\forall t$: $\|f(x,t) - f(y,t)\| \le K\|x - y\|$, then (10) is satisfied with $\delta(\epsilon) = \beta^{-1}(\alpha(\epsilon e^{-KT}))$ which implies global uniform asymptotic stability.

Remark 6: Condition 2 of the theorem may also be replaced by the somewhat weaker condition that $\exists$ a finite T and a $\gamma(\cdot) : \mathbb{R}^+ \to \mathbb{R}$ which is continuous, strictly increasing, and passing through the origin and an open set $U' \subset U$ which contains the origin such that $\forall (x,t) \in U' \backslash \{0\} \times \mathbb{R}$:

$$V(x(t + T(x,t)), t + T(x,t)) - V(x,t) \le -\gamma(\|x\|) < 0 \quad (12)$$

with $T(x,t) \le T$.

Remark 7: If Condition 2 of the theorem is replaced by the condition that $\exists$ a finite T and an open set $U' \subset U$ which contains the origin such that $\forall (x,t) \in U' \backslash \{0\} \times \mathbb{R}$:

$$V(x(t+T), t+T) - V(x,t) \le 0. \quad (13)$$

Then the equilibrium point $x = 0$ of (1) is uniformly stable.

III. Application of the Main Theorem

Considering the linear pendulum with a quasi-periodical friction $f(\frac{t}{\epsilon})$ $(\epsilon > 0)$ where $f(t) = a + \sum_{i=1}^n a_i \sin(\omega_i t + \varphi_i)$, with $n \in \mathbb{N}$, $a_i > 0$, $\omega_i > 0$ and $a > 0$

$$\dot{x} = A\left(\frac{t}{\epsilon}\right)x \quad (14)$$

with

$$A(t) = \begin{pmatrix} 0 & 1 \\ -1 & -f(t) \end{pmatrix}. \quad (15)$$

By applying the main theorem, it will be shown that $\exists \epsilon_1 > 0$ such that $\forall \epsilon \le \epsilon_1$, the pendulum is uniformly asymptotically stable.

Since $a > 0$, the matrix

$$\bar{A} := \begin{pmatrix} 0 & 1 \\ -1 & -a \end{pmatrix} \quad (16)$$

is Hurwitz and there exists a symmetric positive definite matrix Q and a positive constant μ such that

$$\bar{A}^T Q + Q\bar{A} \le -\mu I. \quad (17)$$

Consider the Liapunov function $V(x) = x^T Q x$. The derivative along the solutions of the system is given by

$$\dot{V}(x,t) = x^T \left(A^T\left(\frac{t}{\epsilon}\right)Q + QA\left(\frac{t}{\epsilon}\right)\right)x. \quad (18)$$

If the friction $f(\frac{t}{\epsilon})$ attains positive and negative values, then $\exists t^*$ such that $f(\frac{t^*}{\epsilon}) < 0$. This implies that $A(\frac{t^*}{\epsilon})$ has eigenvalues in the open right half-plane and therefore $A^T(\frac{t^*}{\epsilon})Q + QA(\frac{t^*}{\epsilon})$ has at least one positive eigenvalue. This implies that $\dot{V}(x,t)$ takes positive and negative values which precludes the use of the classical theorem of Liapunov. We will show $\exists$ finite $T > 0$ and $\exists$ γ, defined as in the main theorem, such that $\forall t_0$ and $\forall x_0 \ne 0$:

$$\Delta V(x_0, t_0, T, \epsilon) := V(x(t_0 + \epsilon T)) - V(x_0) \quad (19)$$

$$= \int_{t_0}^{t_0+\epsilon T} \dot{V}(x(t), t)\, dt \le -\gamma(\|x_0\|) < 0. \quad (20)$$

Here, x_0 is the initial state at t_0 and $x(t)$ is the solution of (14) denoted by $x(t) = \Phi(t, t_0)x_0$. By (18), it is possible to rewrite $\Delta V(x_0, t_0, T, \epsilon)$ as

$$x_0^T \left(\int_{t_0}^{t_0+\epsilon T} \Phi^T(t,t_0)\left(A^T\left(\frac{t}{\epsilon}\right)Q + QA\left(\frac{t}{\epsilon}\right)\right)\Phi(t,t_0)\, dt\right)x_0. \quad (21)$$

The system matrix $A(\frac{t}{\epsilon})$ leads to the transition matrix $\Phi(t, t_0)$ which is equal to

$$I + \int_{t_0}^{t} A\left(\frac{\sigma}{\epsilon}\right) d\sigma + \int_{t_0}^{t} A\left(\frac{\sigma_1}{\epsilon}\right) \int_{t_0}^{\sigma_1} A\left(\frac{\sigma_2}{\epsilon}\right) d\sigma_2\, d\sigma_1 + \cdots \quad (22)$$

or

$$\Phi(t, t_0) = I + \Phi'(t, t_0) \quad (23)$$

with an obvious definition of $\Phi'(t, t_0)$. The definition of $f(t)$ implies that $\exists K$ such that $\forall t : \|A(t)\| \le K$ and therefore

$$\|\Phi'(t, t_0)\| \le K(t - t_0) + \frac{K^2}{2}(t - t_0)^2 + \cdots. \quad (24)$$

If $\epsilon T < 1$, then $\forall t \in [t_0, t_0 + \epsilon T]$ one obtains that $t - t_0 < 1$ and $\|\Phi'(t, t_0)\| \le (t - t_0)(e^K - 1)$. By (21) and (23), $\Delta V(x_0, t_0, T, \epsilon)$ can be rewritten as

$$x_0^T \left(\int_{t_0}^{t_0+\epsilon T} \left(A^T\left(\frac{t}{\epsilon}\right)Q + QA\left(\frac{t}{\epsilon}\right)\right) dt\right)x_0 + x_0^T \Delta Q x_0 \quad (25)$$

where the matrix ΔQ is equal to

$$\int_{t_0}^{t_0+\epsilon T} \Phi'^T(t,t_0)\left(A^T\left(\frac{t}{\epsilon}\right)Q + QA\left(\frac{t}{\epsilon}\right)\right) dt \quad (26)$$

$$+ \int_{t_0}^{t_0+\epsilon T} \left(A^T\left(\frac{t}{\epsilon}\right)Q + QA\left(\frac{t}{\epsilon}\right)\right)\Phi'(t,t_0)\, dt \quad (27)$$

$$+ \int_{t_0}^{t_0+\epsilon T} \Phi'^T(t,t_0)\left(A^T\left(\frac{t}{\epsilon}\right)Q + QA\left(\frac{t}{\epsilon}\right)\right)\Phi'(t,t_0)\, dt. \quad (28)$$

By the boundedness of $\|A(t)\|$ and $\|Q\|$ and since $\|\Phi'(t, t_0)\| \le (t - t_0)(e^K - 1)$ if $t \in [t_0, t_0 + \epsilon T]$, one knows that $\exists$ a finite constant c_1 such that $\forall \epsilon$, $\forall t_0$ $(\epsilon T < 1)$

$$\|\Delta Q\| \le c_1(\epsilon T)^2. \quad (29)$$

Taking (29) into account and since

$$A\left(\frac{t}{\epsilon}\right) = \bar{A} + \begin{pmatrix} 0 & 0 \\ 0 & a - f(\frac{t}{\epsilon}) \end{pmatrix} \quad (30)$$

one obtains that

$$\Delta V(x_0, t_0, T, \epsilon) \le x_0^T \left(\int_{t_0}^{t_0+\epsilon T} (\bar{A}^T Q + Q\bar{A})\, dt\right)x_0 \quad (31)$$

$$+ 2\left|\int_{t_0}^{t_0+\epsilon T} \left(a - f\left(\frac{t}{\epsilon}\right)\right) dt\right| \|Q\| \|x_0\|^2 + c_1(\epsilon T)^2 \|x_0\|^2. \quad (32)$$

The specific form of $f(\frac{t}{\epsilon})$ implies that $\exists c_2 > 0$ such that

$$\left|\int_{t_0}^{t_0+\epsilon T} \left(a - f\left(\frac{t}{\epsilon}\right)\right) dt\right| \le \epsilon c_2 \quad (33)$$

and that with the additional restriction $\epsilon T < 1$ one obtains that $\forall t_0$ and $\forall x_0 \ne 0$

$$V(x(t_0 + \epsilon T)) - V(x_0) \le \epsilon(-\mu T + 2c_2\|Q\| + c_1 \epsilon T^2)\|x_0\|^2. \quad (34)$$

Take

$$T > \frac{2c_2\|Q\|}{\mu}$$

such that $-\mu T + 2c_2\|Q\| < 0$ and then take

$$\epsilon < \frac{\mu T - 2c_2\|Q\|}{c_1 T^2}$$

such that $-\mu T + 2c_2\|Q\| + c_1\epsilon T^2 =: -c_3 < 0$. When

$$\epsilon \leq \epsilon_1 := \min\left\{\frac{1}{T}, \frac{\mu T - 2c_2\|Q\|}{c_1 T^2}\right\}$$

with

$$T > \frac{2c_2\|Q\|}{\mu}$$

then $\forall x_0 \neq 0$ and $\forall t_0$ $V(x(t_0+\epsilon T)) - V(x_0) \leq -\epsilon c_3\|x_0\|^2$. With $\gamma(\|x_0\|) := \epsilon c_3\|x_0\|^2$ uniform asymptotic stability is established as a consequence of the main theorem.

Remark 8: The same result can be established by considering $V_I(x) = x_1^2 + x_2^2$ as a Liapunov function. The derivative of this Liapunov function along the solutions of the system is $\dot{V}_I(x(t),t) = -2f(\frac{t}{\epsilon})x_2^2(t)$. If the friction takes positive and negative values, then also $\dot{V}_I(x,t)$ takes positive and negative values precluding the use of the classical theorem of Liapunov. It is possible to prove that $\exists T > 0$, $\exists \epsilon_1 > 0$ and $\exists \gamma > 0$, defined as in the proof, such that $\forall x_0 \neq 0$, $\forall t_0$ and $\forall \epsilon < \epsilon_1$ that

$$V_I(x(t_0+\epsilon T)) - V_I(x_0) \leq -\gamma(\|x_0\|) < 0 \tag{35}$$

which implies uniform asymptotic stability of (14). The proof is straightforward but is omitted.

Remark 9: Notice that the application in the present paper is an example of the averaging result which can be found in Khalil [3, Th. 8.4 and Example 8.13] since the system $\frac{dx}{dt} = A(\frac{t}{\epsilon})x$ is equivalent with $\frac{dx}{d\tau} = \epsilon A(\tau)x$ by setting $\tau = \frac{t}{\epsilon}$. An important feature of the example is the proof technique itself. The result is proved by a Liapunov approach without using a particular state transformation as is usually done in averaging theory.

IV. Discrete-Time Nonlinear Systems

This section is included for completeness. It considers the discrete-time equivalents of the previous results. Consider

$$x_{k+1} = f(x_k, k) \tag{36}$$

with $f: W \times \mathbf{Z} \to \mathbb{R}^n$, W open, $W \subset \mathbb{R}^n$. Let $f(0,k) = 0$ $\forall k \in \mathbf{Z}$ and $0 \in W$. Furthermore, we assume that conditions on (36) are imposed implying existence and uniqueness of its solutions. We single out the local Lipschitz condition: f is locally Lipschitz, i.e., for $\forall x \in W$, $\exists$ a neighborhood $\mathcal{O}(x) \subset W$ such that the restriction $f|_{\mathcal{O}(x)\times\mathbb{Z}} \neq f|_{\mathcal{O}(x)}$ is Lipschitz with Lipschitz function $l_x(k)$ which is bounded $\forall k \in \mathbf{Z}$. Then we have the following analog to Theorem 1.

Theorem 2: Consider a function $V: U \times \mathbf{Z} \to \mathbb{R}$ with $U \subset W$ an open neighborhood of the origin. We assume that the following additional conditions are satisfied.

- *Condition 1:* $V(x,k)$ is positive definite and decrescent; i.e., $V(0,k) = 0$ $\forall k$ and $\forall x \in U: \alpha(\|x\|) \leq V(x,k) \leq \beta(\|x\|)$. The functions $\alpha(\cdot): \mathbb{R}^+ \to \mathbb{R}$ and $\beta(\cdot): \mathbb{R}^+ \to \mathbb{R}$ belong to class $\mathcal{K}$, i.e., they are strictly increasing continuous functions passing through the origin.
- *Condition 2:* $\exists$ finite $m \in \mathbf{Z} > 0$ and a $\gamma(\cdot): \mathbb{R}^+ \to \mathbb{R}$ which belongs to class $\mathcal{K}$, i.e., $\gamma(\cdot)$ is continuous, strictly increasing, and passing through the origin and an open set $U' \subset U$ which contains the origin such that $\forall(x,k) \in U'\backslash\{0\} \times \mathbf{Z}$:

$$V(x(k+m), k+m) - V(x,k) \leq -\gamma(\|x\|) < 0. \tag{37}$$

Then the equilibrium point $x = 0$ of (36) is uniformly asymptotically stable.

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H^∞-Output Feedback Controller Design for Linear Systems with Time-Varying Delayed State

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***Abstract*—This paper considers the H^∞-controller design problem for linear systems with time-varying delays in states. The authors obtain sufficient conditions for the existence of H^∞ controllers in terms of three linear matrix inequalities (LMI's). These sufficient conditions are dependent on the maximum value of the time derivative of time-varying delay. Furthermore, they briefly explain how to construct such controllers from the positive-definite solutions of their LMI's and give an example.**

***Index Terms*—H^∞, LMI, output feedback, time-varying delay.**

I. Introduction

Since time delay is frequently a source of instability and encountered in various engineering systems, the stability problems of time-delay systems have received considerable attention over the last few decades [1]–[12]. There are many publications on solving the stabilization problem of systems with: 1) constant delay [2], [3]; 2) time-varying delay [2], [4]; 3) constant delay and parameter uncertainty [5], [6]; and 4) time-varying delay and parameter uncertainty [7]–[9]. The stability of the closed-loop system in [3], [5], and [6] is independent of time delay, but one in [2], [4], [7]–[9] is dependent on only the maximum value of the time derivative of time-varying delay.

The H^∞-controller design for delay systems is also an interesting problem. In the frequency domain, Lee *et al.* [11] and Choi *et al.* [10] considered memoryless H^∞-state feedback controllers for state-delayed systems and both state- and control-delayed systems, respectively. But when all state variables are not available for the

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