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Autonomous rigid body attitude synchronization

Alain Sarlette, Rodolphe Sepulchre and Naomi Ehrich Leonard

Abstract— **This paper studies some extensions to the decentralized attitude synchronization of identical rigid bodies. Considering fully actuated Euler equations, the communication links between the rigid bodies are limited and the available information is restricted to *relative* orientations and angular velocities. In particular, no leader nor external reference dictates the swarm's behavior. The control laws are derived using two classical approaches of nonlinear control - tracking and energy shaping. This leads to a comparison of two corresponding methods which are currently considered for distributed synchronization - consensus and stabilization of mechanical systems with symmetries.**

I. INTRODUCTION

The distributed synchronization of a set of agents - *i.e.* driving all the agents to a common position and orientation without referring to any leader or external reference - is an ubiquitous task in current engineering problems. Practical applications include autonomous swarm/formation operation ([1]-[23] and references therein), distributed decision making ([24], [25], [26]), and many algorithmical problems involving "dynamical average computations" ([27]). In a modeling framework, the understanding of swarm behavior has also led to many important studies ([12], [28], [29]). Synchronization on non-Euclidean manifolds raises particular questions ([30], [31]), but also appears in several applications. Beyond the circle ([9], [10], [11], [32]), the most important example is perhaps the group $SO(3)$, representing the orientations of 3-dimensional rigid bodies ([3], [6], [7], [33], [34], [35]). For example, attitude synchronization is required for modern space mission concepts involving multiple satellites flying in formation ([19]-[23],[36]-[39]).

The present paper addresses the distributed synchronization problem on $SO(3)$ in a consensus and dynamical setting. In particular, the goal is to find control torques that asymptotically drive a swarm of fully actuated rigid bodies towards the same orientation under two constraints. First, an imposed interconnection graph limits the information exchange between the agents. Second, no leader nor external reference is allowed to dictate the swarm's behavior: the agents are strictly identical and must synchronize without any external help. This second constraint expresses the fully *autonomous and distributed character* of the problem.

A first viewpoint on rigid body synchronization comes from the field of control of mechanical systems. Most of the studies from this field rely on a common external reference which must be tracked by the agents or introduce a leader in the swarm ([2], [22], [23], [33]-[36]). Some of those authors also noticed singular behavior due to their use of the non-unique quaternion representation. Instead, the present paper works directly on the manifold $SO(3)$ and avoids leaders or external references. This is in the line of the work in [1], [3]-[7], [40]. However, the latter only considers specifically fixed interconnection topologies. Moreover, the presented stability results are not asymptotic, unless an external reference is introduced. In summary, none of this work achieves asymptotic synchronization in a fully autonomous and distributed way.

Another class of previous work in the area concerns *consensus algorithms*. This approach works perfectly autonomously: the focus lies on the absence of external reference (hence the need for consensus) and on imposed communication constraints. However, working in a computational or task-planning framework, the agents are often modeled as first-order integrators, disregarding the full system dynamics (in particular, for attitude control, the nonlinear Euler equations). Most existing consensus results are valid for Euclidean state spaces ([12], [16], [17], [25], [26], [41]-[44]), but recent work also considers non-Euclidean spaces ([8]-[11], [18], [27], [32], [45]), mainly the circle. The work in [45] about consensus on homogeneous manifolds allows to apply the consensus approach to $SO(3)$.

In the present paper, the consensus approach and the mechanical approach are brought together. To the best of our knowledge, these are the first results that (i) achieve *asymptotic* synchronization (ii) of a mechanical system that evolves on the *non-Euclidean manifold* $SO(3)$, and explicitly consider both (iii) the *autonomous, distributed character* of the problem and (iv) the Euler equations describing the system dynamics.

In both the consensus and the mechanical approaches, synchronization can be viewed as a distributed optimization task. The difference lies in the way this optimization problem is solved. For the "consensus tracking" point of view, a consensus algorithm is used at a task planning level to define desired trajectories which are tracked by the individual agents. The mechanical point of view applies the "energy shaping" method: the cost function is used as an artificial potential and properly designed artificial dissipation asymptotically stabilizes the synchronized state as a minimum of the artificial potential ([1], [3], [6], [7]).

The paper is organized as follows. The distributed attitude synchronization problem is defined in Section II. Section III highlights the similarities in previous work on consensus and energy shaping. Section IV introduces the indirect

A. Sarlette and R. Sepulchre are with the Department of Electrical Engineering and Computer Science (Montefiore Institute, B28), University of Liège, 4000 Liège, Belgium. alain.sarlette@ulg.ac.be, r.sepulchre@ulg.ac.be

N.E. Leonard is with the MAE Department, Princeton University, NJ 08544, USA. naomi@princeton.edu

or consensus tracking approach, leading to new control laws. Section V presents new results obtained with the introduction of dissipation in the energy shaping method. Proofs are omitted because of space limitations.

II. PROBLEM SETTING

Consider a swarm of n identical rigid bodies, called *agents* in the following. An orthonormal reference frame $\mathbf{B_k}$ is attached in the same way to each body k such that the moment of inertia matrix in $\mathbf{B_k}$ is $J = \text{diag}(J_1, J_2, J_3)$ with $J_1 > J_2 > J_3$. The orientation of $\mathbf{B_k}$ with respect to an inertial frame $\mathbf{A}$ is given by the rotation matrix Q_k such that a vector $\mathbf{v}$ with components v in $\mathbf{A}$ has components $Q_k^T v$ in $\mathbf{B_k}$.

The variation with respect to time of the orientation of $\mathbf{B_k}$ in $\mathbf{A}$ is characterized by the kinematic equation

$$\tfrac{d}{dt} Q_k = Q_k [\omega_k]^\wedge \tag{1}$$

where ω_k denotes the *angular velocity* of agent k expressed in $\mathbf{B_k}$ and $[\cdot]^\wedge : \mathbb{R}^3 \to \mathfrak{so}(3)$ denotes the skew-symmetric matrix implementing the cross-product $[x]^\wedge y = x \times y \ \forall y$. For $x = (x_1\ x_2\ x_3)^T$ this implies

$$[x]^\wedge = \begin{pmatrix} 0 & -x_3 & x_2 \\ x_3 & 0 & -x_1 \\ -x_2 & x_1 & 0 \end{pmatrix} .$$

The inverse of $[\cdot]^\wedge$ is denoted $[\cdot]^\vee : \mathfrak{so}(3) \to \mathbb{R}^3$. For example, $\omega_k = \left[Q_k^T \tfrac{d}{dt} Q_k\right]^\vee$.

The dynamics of agent k are governed by

$$J \tfrac{d}{dt}\omega_k = (J\omega_k) \times \omega_k + \tau_k \tag{2}$$

(Euler equations) where the torque τ_k is expressed in $\mathbf{B_k}$. The objective of the present paper is to design control inputs τ_k coupling the n agents such that the n systems of equations (1)-(2), $k = 1, ...n$, drive the swarm to a *synchronized state* where $Q_k = Q_j \ \forall k, j$, whatever their absolute orientation and absolute angular velocity may be. As noted in [1], [40], only the absolute orientation can be factored out of the state space. Indeed, the dynamics (2) are invariant w.r.t. a fixed rotation Q of all the agents, but not w.r.t. any synchronized motion since the angular velocity ω_k appears explicitly.

The agents are assumed to be fully actuated. However, they have access to limited information. In particular, each agent k only gets information from a subset of the other agents. This is denoted by $j \rightsquigarrow k$ ("j is a neighbor of ≡ sends information to k"). The collected communication links form a directed graph G. G is *undirected* if $k \rightsquigarrow j$ whenever $j \rightsquigarrow k$. The *associated undirected graph* of a directed graph G simply replaces all the directed links of G by bidirectional links. It is assumed that agent k gets the following information from each of its neighbors j:

- its relative orientation $Q_k^T Q_j$;
- its relative angular velocity $Q_k^T(Q_j\omega_j - Q_k\omega_k)$;
- possibly a set of scalar auxiliary variables X_j.

In addition, k may have to measure $\omega_k = \left[Q_k^T \tfrac{d}{dt} Q_k\right]^\vee$, its own angular velocity w.r.t. $\mathbf{A}$ and expressed in $\mathbf{B_k}$ (in the remainder of the paper this is called the *absolute* or *inertial* angular velocity of k, in opposition to its *relative* angular velocity which characterizes its motion w.r.t. another agent). However, it never knows its absolute orientation Q_k. Note that the measurement of ω_k does not necessarily involve an external reference but can be retrieved by an isolated agent equipped with an inertial device (gyroscope). This is consistent with the symmetries of the dynamics.

III. CONSENSUS COST FUNCTION

The goal of the present section is mainly to highlight the similarities of the consensus and energy shaping approaches to attitude synchronization. Many specificities of both approaches are therefore neglected, most notably the important work on mechanical symmetries and reduction described among others in [1], [6], [7], [40]. Notations and formulations have also been adapted, for the sake of simplicity and at the cost of generality. Among others, in an effort to design globally valid algorithms, an inherent geometric representation of $SO(3)$ is favored and the more popular quaternion representation ([2], [23], [36]) is ignored.

Consider the function

$$P_G(Q_1, ..., Q_n) = \frac{1}{2n^2} \sum_{k=1}^{n} \sum_{j \rightsquigarrow k} \text{trace}\left(Q_j^T Q_k\right) .$$

In a mechanical/energy-shaping framework, the function P_G itself was already introduced in [1], [34], [40], [3]. It can also be directly derived from the *chordal distance* $d(Q_j, Q_k) = \sqrt{\text{trace}(I_3 - Q_j^T Q_k)}$ introduced in [31] to represent the agreement between interconnected agents in a consensus/algorithmical framework. In fact, it is shown in [45] how this function arises as a special case of consensus functions on compact connected homogeneous manifolds. One easily checks that

$$\text{trace}\left(Q_j^T Q_k\right) = 1 + 2\cos(\theta)$$

where $\theta \in [0, \pi]$ is the rotation angle of the single-axis rotation between Q_j and Q_k (see Euler's rotation theorem) and represents the canonical geodesic distance from Q_j to Q_k $\in SO(3)$. P_G is thus a measure of attitude synchronization in the swarm. Consequently, if G is connected, synchronization is the unique global maximum of P_G; unless otherwise specified, the present paper always assumes that G is connected. Because $SO(3)$ is a non-Euclidean manifold, in general, P_G may have local maxima. The following proposition identifies particular situations where the synchronized state is the only local maximum of P_G.

Prop. 1: *If the undirected graph G associated to the agent interconnections is a complete graph or a tree, then attitude synchronization is the only maximum of P_G.*

In an energy-shaping framework, P_G is used to build an artificial potential σP_G, $\sigma < 0$, whose global minimum corresponds to the synchronized state ([1], [3], [6], [7]). This leads to the following basic control torque, which only uses information on relative orientations of the agents in a fixed, undirected interconnection graph G ([3], [7], [1], [6]):

$$\tau_k^{(S)} = -\left[Q_k^T \tfrac{\partial\, \sigma P_G}{\partial Q_k}\right]^\vee = \tfrac{-\sigma}{2n^2}\,[\sum_{j\rightsquigarrow k}(Q_k^T Q_j - Q_j^T Q_k)]^\vee. \quad (3)$$

Using different energy-related methods, [1] proves that synchronization with zero velocity is a stable equilibrium of (1)-(2) with the control (3), and [3], [6], [7] extend this stability result to the case where the rigid bodies rotate together about their aligned short axis. However, the Hamiltonian of the system (consisting of the kinetic energy and the artificial potential, such that (3) is accounted for by the artificial potential) is conserved, such that some form of dissipation must be added to obtain *asymptotic* stability. This is left as an open question in [3] and solved with the help of an external reference frame in [6], [7]. In [1], it is suggested to use external dissipation (drag) to obtain asymptotic stability. Dissipation without any external reference is considered in the present paper, see Section V. Due to the derivation of the control torques from the potential σP_G, the energy shaping approach is restricted to fixed undirected interconnection graphs G and sensitive to the local minima of σP_G imposed by G.

This limitation can be removed in the consensus framework. However, the consensus algorithms do not consider the dynamics of the system. Instead, they define desired kinematic trajectories to maximize P_G, assuming that the angular velocity ω_k is a direct control input. When G is fixed and undirected, the system is simply steered along the gradient of P_G, leading to the desired trajectory

$$Q_k^T \tfrac{d}{dt} Q_k = [\omega_k^{(d)}]^\wedge = \tfrac{\alpha_k}{2n^2} \sum_{j\rightsquigarrow k}\left(Q_k^T Q_j - Q_j^T Q_k\right) \quad (4)$$

where $\alpha_k > 0$. Note the similarity between this desired rotational velocity and the control torque (3). When G is directed and/or time-varying, (4) no longer defines a gradient system and, on non-Euclidean manifolds, solutions do not always converge to synchronization. In this case, the approach first proposed in [32] for the circle and generalized to homogeneous manifolds in [45] is to equip each agent with a consensus estimator (see also [27], [8], [10]). For $SO(3)$, this auxiliary variable X_k is a 3×3-array of numbers that interconnected agents communicate. Defining $N_k = Q_k X_k$, the equations computed by the agents in their local frames $\mathbf{B}_k$ can be rewritten as ([45])

$$\tfrac{d}{dt} N_k \quad = \quad \alpha_k^{(1)} \sum_{j\rightsquigarrow k}(N_j - N_k) \quad (5)$$

$$Q_k^T \tfrac{d}{dt} Q_k = [\omega_k^{(d)}]^\wedge \quad = \quad \tfrac{\alpha_k^{(2)}}{2}(Q_k^T N_k - N_k^T Q_k) \quad (6)$$

where the first equation is expressed in inertial frame $\mathbf{A}$ and $\alpha_k^{(1)}, \alpha_k^{(2)} > 0$. The first equation (5) is a classical linear consensus algorithm in Euclidean space, leading to synchronization of the N_k under the following assumption (see [26], [41], [42]).

Ass. 1 [42]: *The interconnection graph $G(t)$ is piecewise continuous in time. Moreover, define the graph $G_{\delta,T}(t)$ to contain all the interconnections that are found in $G(t)$ for at least δ seconds during the time interval $[t, t+T]$. Then it is assumed that there are time constants $\delta > 0$, $T > 0$ and an agent k such that for all t, in the graph $G_{\delta,T}(t)$ there is a path from agent k to all the other agents. This property is known as* uniform connectedness [42].

The second equation simply makes Q_k track the projection of N_k on $SO(3)$. The latter can be computed in closed form and is generically unique, so that the kinematic algorithm generically converges to synchronization when Assumption 1 is satisfied. Note that, by using a consensus algorithm in Euclidean space, the problem of local maxima is circumvented.

IV. CONSENSUS TRACKING

The consensus algorithms (4) and (5)-(6) directly assign a velocity to each agent k at each time instant t. A second step is required to obtain dynamical algorithms, *i.e.* explicit expressions for the control torques τ_k of (2). The new contribution of this section is to briefly discuss this link from consensus to dynamics.

The knowledge of an individual's own absolute angular velocity ω_k is unavoidable for the control laws of the present section. With this information, it is rather obvious to make the agents individually track a desired angular velocity field $\omega_k^{(d)}$ defined by a consensus algorithm. The simplest control strategy based on (5) defines the desired *orientation* $Q_k^{(d)}$ to be the projection of N_k on $SO(3)$; equation (6) is replaced by a dynamical tracking algorithm on $SO(3)$. Since the projection process from $\mathbb{R}^{3\times 3}$ to $SO(3)$ presents a discontinuity when X_k is singular, $Q_k^{(d)}$ is not necessarily continuous. However, it is unimportant to track the transient trajectory: the only objective is to synchronize the rigid bodies towards the final consensus value of N_k. In this setting, it might even seem useless to move the rigid bodies before the auxiliary variables approach a consensus situation. Moving the rigid bodies into the desired attitude after the agents have reached consensus on N_k would just require a global attitude *stabilization* controller for each agent. Algorithms for attitude tracking or stabilization may be found among others in [19], [33]-[36], [46]. Tracking approaches to attitude coordination can also be found in [2], [22], [23], though the presence of a common external reference is necessary for their results.

The following explicitly considers some control torques for a consensus tracking synchronization strategy based on (5). Both (4) and (6) impose zero angular velocity when synchronization is achieved. To be more general, a common constant (in body frame) angular velocity ω_0 is imposed to the synchronized swarm (ω_0 may *e.g.* result from a consensus algorithm in $\mathbb{R}^3$). Therefore, the desired orientation for Q_k becomes $N_k e^{t[\omega_0]^\wedge}$ and the desired angular velocity becomes

$$\omega_k^{(d)} = \omega_0 + \tfrac{\alpha_k^{(2)}}{2}\,[\,Q_k^T N_k e^{t[\omega_0]^\wedge} - e^{-t[\omega_0]^\wedge} N_k^T Q_k\,]^\vee\,. \quad (7)$$

In order to drive ω_k to $\omega_k^{(d)}$, an exponential evolution

$$\tfrac{d}{dt}(\omega_k - \omega_k^{(d)}) = -\beta_k(\omega_k - \omega_k^{(d)}) \qquad (8)$$

is imposed, where $\beta_k > 0$ and $\frac{d}{dt}\omega_k^{(d)}$ is deduced from (7) and (5). Given the form of (2), the torque

$$\tau_k = J\tfrac{d}{dt}\omega_k^{(d)} - \beta_k(\omega_k - \omega_k^{(d)}) - (J\omega_k) \times \omega_k \qquad (9)$$

achieves (8) by using available information, *i.e.* auxiliary variables, relative orientations and relative angular velocities of interconnected agents and one's own absolute angular velocity.

Prop. 2: *Consider a swarm of agents applying the control torque (9) where $\omega_k^{(d)}$ is defined by (7), in combination with (an equivalent in body coordinates of) the consensus algorithm (5). If the interconnection graph satisfies Assumption 1 and the initial values of N_k are randomly chosen in $\mathbb{R}^{3\times 3}$, then the only stable limit set for $t \to +\infty$ is synchronization of the orientations with $\omega_k = \omega_0\ \forall k$.*

Unstable situations include some agents being turned by exactly 180 degrees w.r.t. the common $Q^{(d)}$, and the rare cases where the N_k converge to a matrix for which $Q_k^{(d)}$ is not uniquely defined. Limit cycles are not expected.

The control torque (9) includes a term that exactly cancels the free rigid body dynamics. This is characteristic of the "computed torque" method and requires perfect knowledge of the inertia matrix J. An alternative is to dominate the natural dynamics using a high-gain method. If β_k is large enough, the control torque (9) can be reduced to

$$\tau_k = J\tfrac{d}{dt}\omega_k^{(d)} - \beta_k(\omega_k - \omega_k^{(d)})\ . \qquad (10)$$

Prop. 3: *Consider a swarm of agents applying the control torque (10) where $\omega_k^{(d)}$ is defined by (7), in combination with (an equivalent in body coordinates of) the consensus algorithm (5). Moreover, assume that the interconnection graph satisfies Assumption 1, the initial values of N_k are randomly chosen in $\mathbb{R}^{3\times 3}$ and ω_0 is aligned with the principal axis of the rigid bodies corresponding to J_i. If, $\forall k$,*

$$\beta_k > \tfrac{1}{J_3}\left(\tfrac{3}{\sqrt{2}}\alpha_k J_1 \|N_k\|_F + \tfrac{3J_1+J_i}{2}\|\omega_0\|\right) \qquad (11)$$

where $\|\cdot\|_F$ denotes the Frobenius norm, then the only stable limit set for $t \to +\infty$ is synchronization of the orientations with $\omega_k = \omega_0\ \forall k$.

Remarks:

1) The values of $\|N_k\|_F$ are ensured to be bounded because, for whatever interconnection graph and at any time, the values of $N_k(t)$ lie in the convex hull of the initial values $\{N_1(0), N_2(0), ...N_n(0)\}$.
2) The values of α_k and β_k in the algorithms of the present section may be smoothly varied by each agent above arbitrarily fixed lower bounds $\alpha_{min} > 0$ and $\beta_{min} > 0$. This enables adaptation to constraints of the system. For example, one may choose α_k inversely proportional to $\|N_k\|_F$ in order to satisfy (11).
3) Proposition 3 allows to impose a rotation ω_0 about a principal axis only. This is probably the most useful case in practice, as maintaining other motions would require persistent control torques.

V. ENERGY SHAPING

The present section considers extensions of the energy shaping approach of [1], [3], [6], [7]: introducing new *dissipative* terms, the goal is to obtain *asymptotic* synchronization with control torques satisfying the assumptions about available information made in Section II. The energy shaping approach leads to simpler and arguably more robust control laws. Moreover, the basic control torque (3) can be computed without requiring any information about angular velocities; those will only appear through the dissipation. As a consequence, (3) imposes no restrictions on the final motion of the synchronized agents; the set of possible motions will only be reduced according to the symmetries of the inherent dynamics and the dissipation term. At the end of this section, a locally synchronizing control torque that can be implemented *without absolute angular velocity (ω_k) measurements* is presented. A current limitation of energy shaping is that the interconnection graph G must be fixed and undirected (and connected).

The dissipative torque to be designed is denoted $\tau_k^{(D)}$, such that $\tau_k = \tau_k^{(S)} + \tau_k^{(D)}$ with $\tau_k^{(S)}$ defined in (3). The total energy (including the artificial potential) of the swarm

$$H = \tfrac{1}{2}\sum_{k=1}^{n}\left(\omega_k^T J\omega_k + \tfrac{\sigma}{2n^2}\sum_{j \rightsquigarrow k}\mathrm{trace}(Q_k^T Q_j)\right)$$

evolves as $\frac{d}{dt}H = \sum_k \omega_k^T \tau_k^{(D)}$.

A. Dissipation w.r.t. inertial space

Introducing dissipation on the motion of each individual agent is admissible if each agent measures its own angular velocity. This was already suggested in [1].

Prop. 4: *The control torque $\tau_k = \tau_k^{(S)} - b_k\omega_k$ with $b_k > 0$ and $\tau_k^{(S)}$ defined in (3), drives the swarm towards an equilibrium set where $\omega_k = 0\ \forall k$ and the Q_k are at a critical point of P_G. The only stable equilibria are the maxima of P_G.*

Proposition 4 uses a simplified control torque with respect to Propositions 2 and 3. In particular, no exchange of angular velocities is needed, the free rigid body dynamics are not counteracted and there is no condition on the strength of the control torques. However, the convergence result is significantly weaker. Indeed, it is restricted to fixed, undirected graphs and, except for special graphs like trees or complete graphs, does not exclude locally stable equilibria that are different from synchronization. Moreover, the introduction of dissipation in inertial space always stabilizes the agents at rest ($\omega_0 = 0$ with respect to Propositions 2 and 3). The first issue is inherent to the energy shaping approach based on P_G. The second issue is addressed in the following, where "inter-agent dissipation" is considered in order to synchronize the agents in a moving situation.

B. Dissipation in shape space

A more elegant way to introduce dissipation in the context of relative motions is through the *relative* angular velocities. This replaces the measurement of absolute angular velocities ω_k by relative angular velocities of interconnected agents, such that the agents can implement their control torque without any absolute information about their own state. The resulting control torque is

$$\tau_k = \tau_k^{(S)} - b \sum_{j \rightsquigarrow k} (\omega_k - Q_k^T Q_j \omega_j) \qquad (12)$$

with $b > 0$. A fundamental property of (12) is the fact that the torques $Q_k \tau_k$ in inertial space $\mathbf{A}$ sum to zero, such that the total angular momentum of the swarm $\sum_k Q_k J \omega_k$ is conserved.

The control (12) always asymptotically leads to *angular velocity synchronization, i.e.* $Q_k \omega_k = Q_j \omega_j \ \forall k, j$. This means that asymptotically, the relative orientations $Q_k^T Q_j$ in the swarm are constant. As a consequence, the control torques τ_k are all constant as well. However, the control law does not necessarily lead to attitude synchonization. A simple counterexample with just 2 rigid bodies indicates that asymptotic orientation synchronization - even locally - requires an additional assumption on the relative strength of the artificial potential with respect to the kinetic energy.

Prop. 5: *Assume that G is connected. The control torque (12) where* $\tau_k^{(S)}$ *is defined in (3), drives the swarm towards an invariant set under (3) with synchronized angular velocities* $Q_k \omega_k$ *(and hence fixed relative orientations* $Q_k^T Q_j$*). Moreover, for every set of initial angular velocities, there exists a constant* $\sigma^* < 0$ *(actually depending on* n*,* J*,* G *and the initial* kinetic energy $K(0)$ *only) such that when* $|\sigma| > |\sigma^*|$*, the orientations* Q_k *of the agents locally asymptotically synchronize.*

Sketch of the proof: The fact that the angular velocities $Q_k \omega_k$ synchronize readily appears by using H as a Lyapunov function. For the synchronization of the orientations Q_k, the bound $|\sigma|^*$ serves two purposes. First, it is shown that given a neighborhood W of synchronization, there exist a value $|\sigma_1|$ and a neighborhood U of synchronization such that starting in U imposes staying in W for $|\sigma| > |\sigma_1|$. Then it is shown, using linearization, that there exists a value $|\sigma_2|$ such that for $|\sigma| > |\sigma_2|$, synchronization is a locally unique solution of (1)-(2)-(3) with identical angular velocities $Q_k \omega_k$. Choosing W such that this linear/local result is valid inside W allows us to conclude that solutions starting in U converge to synchronization for $|\sigma| > \max(|\sigma_1|, |\sigma_2|)$. The conservation of total angular momentum plays a central role in the proof. $\triangle$

In the reduced space $(TSO(3))^n/TSO(3)$ of relative orientations and relative angular velocities, the statement about orientation synchronization is equivalent to local asymptotic stability of the isolated equilibrium $Q_k^T Q_j = I_3, \ \ Q_k^T Q_j \omega_j - \omega_k = 0 \ \forall k, j$. The absolute angular velocity is then an external variable inducing time-varying dynamics. The bound $|\sigma| > |\sigma^*(K(0))|$ for synchronization is non-uniform with respect to the (initial) absolute angular velocity.

Remarks:

1) The sum appearing in $\tau_k^{(D)}$ of (12) need not consider the same interconnection graph G as for $\tau_k^{(S)}$. It is just a natural assumption in the present context.
2) The control (12) vanishes when the agents are all synchronized. Hence any synchronized free rigid body motion is an equilibrium in $(TSO(3))^n/TSO(3)$ for (1)-(2) with input (12).
3) The design of relative dissipation in mechanical systems is also addressed in [47], with a different approach.

VI. CONCLUSION

The goal of the present work was to explore some issues arising when synchronization algorithms on *non-Euclidean manifolds* are designed at a *dynamical level* with *limited inter-agent communications* and *relative orientation and angular velocity measurements* only.

In contrast with the simplified kinematic description ([8], [10], [11],...), the rigid body *dynamics* do not retain full symmetry w.r.t. any synchronized motion of the agents ([1], [40]). Therefore, a tracking approach ([2], [33]-[36]) requires each individual to know its own angular velocity.

For fixed undirected interconnection graphs, the energy shaping approach can be applied ([3], [6], [7]). The main issue here is to design artificial dissipation without referring to absolute angular velocities.

An interesting topic for future research is to try to combine the energy shaping control (12) with estimator variables as in (5)-(6) to achieve *global* attitude synchronization for potentially *directed and time-varying* interconnection graphs and using only *relative* angular velocities.

VII. ACKNOWLEDGMENTS

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