Autonomous rigid body attitude synchronization ⋆

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Abstract

Control laws to synchronize attitudes in a swarm of fully actuated rigid bodies, in the absence of a common reference attitude or hierarchy in the swarm, are proposed in [31, 19]. The present paper studies two separate extensions with the same energy shaping approach: (i) locally synchronizing the rigid bodies' attitudes, but without restricting their final motion and (ii) relaxing the communication topology from undirected, fixed and connected to directed, varying and uniformly connected. The specific strategies that must be developed for these extensions illustrate the limitations of attitude control with reduced information.

Key words: Attitude control, Autonomous swarms, Consensus, Cooperative control, Energy shaping

1 Introduction

Coordination is a common requirement in applications involving robotic swarms or formations [5, 14, 4, 30, 9, 29]; understanding basic coordination mechanisms is the subject of ongoing research [5, 24, 32]. Controlling a swarm of three-dimensional rigid bodies such that their orientations become asymptotically equal is called *attitude synchronization*. Its main use is in satellite formations [7, 16, 35, 12, 3, 11], e.g. for the Darwin mission, a space interferometer project under study by NASA and ESA, or for on-orbit assembly [7, 16]. Operational requirements focus on *accuracy* near equilibrium [1]. The present paper focuses on *convergence* from arbitrary initial orientations with *limited information*. This is probably most relevant for deployment or recovery. Each rigid body is assumed fully actuated and called an *agent*.

A first main constraint limits communication links among agents. Under this constraint, [35, 12] consider attitude synchronization coupled to the tracking of a common external reference. Alternatively, [3, 11] synchronize attitudes in a leader-follower approach.

The qualifier "autonomous" refers to a second main constraint in the present paper: there is no hierarchy in the swarm and no external reference tracking. Autonomous operation in multi-agent systems is well motivated. It can increase robustness, since synchronization does not rely on permanent communication of a common reference, nor on the health of a potential leader. Also, it can stabilize the formation without interfering with its absolute motion. When orientation in inertial frame is not relevant (e.g. in assembly), this additional freedom may lower costs; in other cases, it builds a swarm that behaves more like a single body than a set of individual agents. The autonomous setting relies on the global Lie group structure of orientation manifold $SO(3)$. Therefore the popular unitary quaternion representation, containing two elements for each point of $SO(3)$, cannot be used.

Control laws for autonomous attitude synchronization are designed in [31, 19] with energy shaping. The present paper extends them in two separate ways.

1. In [31, 19], the dissipative control term, based on angular velocities, imposes the final motion of the swarm. By only using *relative* angular velocities between agents, the present paper obtains a swarm behaving like a single rigid body: any synchronized free rigid body motion is a solution for the controlled swarm.
2. The results in [31, 19] are valid for fixed, undirected communication topologies. Inspired by consensus strategies on compact Lie groups [26, 28], the present paper uses auxiliary variables to allow directed and time-varying communication topologies.

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Both extensions illustrate difficulties encountered when reducing available information. The limitations of "relative dissipation" show the difficulty to control a system whose dynamics just have configuration symmetry with a controller that has configuration and velocity symmetry; this illustrates the relevance of formal *reduction techniques* [6]. With limited communication, it is well-known that for more than local synchronization, the non-convexity of compact Lie groups requires new strategies as compared to simple vector space algorithms [18].

The paper is organized as follows. Section 2 formalizes the problem and reviews the main result of [31, 19]. Section 3 presents extension 1 and Section 4 extension 2.

2 Problem setting and previous results

Orientations of N rigid bodies with respect to an inertial frame are represented by rotation matrices $Q_k \in SO(3)$, $k = 1...N$. Their motion follows Euler's equations

$$Q_k^T \tfrac{d}{dt} Q_k = [\omega_k]^\wedge \qquad (1)$$
$$J_k \tfrac{d}{dt}\omega_k = [J_k\omega_k]^\wedge \omega_k + \tau_k \qquad (2)$$

where $J_k = \text{diag}(J_{k1}, J_{k2}, J_{k3})$ is the moment of inertia matrix of agent k, $\omega_k \in \mathbb{R}^3$ its angular velocity and $\tau_k \in \mathbb{R}^3$ its control torque; these are all expressed in body frame and $J_{k1} \geq J_{k2} \geq J_{k3}$ without loss of generality. Torque and velocity in inertial frame are $Q_k\tau_k$ and $Q_k\omega_k$. Matrix transpose is denoted $\cdot^T$, and $[\cdot]^\wedge$ is defined by

$$a = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} \in \mathbb{R}^3 \;\leftrightarrow\; [a]^\wedge = \begin{pmatrix} 0 & -a_3 & a_2 \\ a_3 & 0 & -a_1 \\ -a_2 & a_1 & 0 \end{pmatrix} \in \mathfrak{so}(3)$$

such that $a \times b = [a]^\wedge b$, $\forall a, b \in \mathbb{R}^3$ with $\times$ the vector product. The inverse of $[\cdot]^\wedge$ is denoted $[\cdot]^\vee : \mathfrak{so}(3) \to \mathbb{R}^3$. The objective is to design τ_k, $k = 1...N$ such that $Q_1 = ... = Q_N$ asymptotically, under two main constraints.

1. (communication) Communication links among agents are restricted to the edges of a *communication graph* G; "j sends information to k" is denoted $j \rightsquigarrow k$.
2. (autonomy) Agents use no external reference; available information is expressed relative to body frame.

References [31, 19] solve this problem with *energy shaping*: "shape" the potential and kinetic energy of a system to make the desired state a stable equilibrium; control torques implement the "shaped" energy. Early work uses artificial potentials for robotic navigation and obstacle avoidance [10, 25]. Spacecraft control uses potential [15] and kinetic [2] energy shaping. Potential shaping is used in [13] for stabilization of rigid bodies in $SE(3)$. Energy shaping is used for synchronization of mechanical system networks in [20] and applied to networks on $SO(3)$ and $SE(3)$ in [31, 19, 6]. Kinetic energy shaping can transform any principal axis into the short axis [31, 19]; this part is ignored here for simplification. $d_{jk} := \sqrt{3 - \text{trace}(Q_k^T Q_j)}$ characterizes the distance between Q_k and Q_j, and $(d_{jk})^2$ is smooth. Therefore [31, 19] use artificial potential

$$V = \tfrac{\sigma}{2} \textstyle\sum_k \sum_{j \rightsquigarrow k} \text{trace}(Q_k^T Q_j) \;, \qquad \sigma < 0 \qquad (3)$$

whose global minimum is attitude synchronization. V can have local minima when G is not a tree or complete graph [26]. For G undirected (i.e. $j \rightsquigarrow k \Leftrightarrow k \rightsquigarrow j$), the conservative control torques only depend on the relative orientations $Q_k^T Q_j$ of agents $j \rightsquigarrow k$ with respect to k:

$$\tau_k^{(P)} = -[\text{grad}_{Q_k}(V)]^\vee = -\sigma \sum_{j \rightsquigarrow k} [Q_k^T Q_j - Q_j^T Q_k]^\vee . \quad (4)$$

Write $\tau_k = \tau_k^{(P)} + \tau_k^{(D)}$, energy $H = T + V$ with kinetic energy $T = \sum_k T_k = \sum_k \frac{1}{2}\omega_k^T J_k \omega_k$, and angular momentum $M = \sum_k Q_k J_k \omega_k$. $\frac{d}{dt}H = \sum_k \omega_k^T \tau_k^{(D)}$ and $\frac{d}{dt}M = \sum_k Q_k \tau_k$. When $\tau_k^{(D)} = 0$, attitude synchronization with rotation around the short axis is (Lyapunov) stable. *Asymptotic* stability requires $\tau_k^{(D)}$ to decrease H. [19] exponentially stabilize this rotation when V contains an additional term *aligning the short axis with a specific direction in inertial space*; this does not satisfy autonomy. An alternative torque satisfying autonomy is

$$\text{with } \tau_k^{(P)} \text{ from (4)}, \qquad \tau_k^{(D)} = -\gamma\, \omega_k \;, \quad \gamma > 0 \qquad (5)$$

as in [23], for which H decreases till $\omega_k = 0\ \forall k$. This asymptotically stabilizes attitude synchronization with zero velocity.

3 Extension 1: relative angular velocities

Dissipation (5), although it preserves symmetry with respect to *orientation* of the agents (autonomy constraint), imposes their *motion*. A dissipative term preserving motion symmetry should drive agent velocities *towards each other* instead of towards 0, comparing $Q_k\omega_k$ to the $Q_j\omega_j$ (inertial frame), or equivalently ω_k to the $Q_k^T Q_j \omega_j$ (body frame). The corresponding torque is

$$\tau_k^{(D)} = \gamma \sum_{j \rightsquigarrow k} (Q_k^T Q_j \omega_j - \omega_k) \;, \qquad \gamma > 0 \, . \qquad (6)$$

Attitude synchronization is more difficult with (6) than with (5) because the τ_k only influence *relative* velocities, while the rigid body dynamics still depend on ω_k; indeed, the nonlinearity of (2) with respect to ω_k cannot be reduced. The study of mechanical systems that are invariant with respect to configuration symmetries, but still depend on the associated velocities, is the subject of *reduction techniques* – see [6] for a discussion about $SO(3)$

and $SE(3)$. The following result illustrates typical difficulties caused by the remaining velocity in the dynamics. The proof also illustrates the difficulty to obtain results without resorting to formal reduction techniques.

Theorem 1 *Consider G fixed, undirected and connected and control $\tau_k = \tau_k^{(P)} + \tau_k^{(D)}$ with $\tau_k^{(D)}$ defined by (6) and $\tau_k^{(P)} = -[\mathrm{grad}_{Q_k}(V)]^\vee$, where V is a bounded potential.*

(a) Regardless of V, for any initial conditions, velocities in inertial frame $Q_k\omega_k$ asymptotically synchronize.
(b) For identical rigid bodies ($J_k = J\ \forall k$) and V defined by (3), angular momentum M is conserved. Given M_{max}, there exists $\sigma^* < 0$ (depending on N, J, G and M_{max}) such that for $|\sigma| > |\sigma^*|$, the set of free rigid body motions with synchronized attitudes $Q_k(t) = Q_j(t)\ \forall j,k$ and $\|M\| \in (0, M_{max})$ is locally asymptotically stable.

PROOF. (a) $\frac{d}{dt}H = \sum_k \omega_k^T \tau_k^{(D)}$
$= \gamma \sum_k \sum_{j \rightsquigarrow k} \omega_k^T (Q_k^T Q_j \omega_j - \omega_k)$
$= \gamma \sum_k \sum_{j \rightsquigarrow k} (Q_k\omega_k)^T (Q_j\omega_j - Q_k\omega_k)$
$= -\gamma\, (\Omega^a)^T (L \otimes I_3) \Omega^a$
where Ω^a is the $3N$-vector containing all $Q_k\omega_k$ and $L \otimes I_3$ is the Kronecker product of the Laplacian[1] L of G with the 3×3 identity matrix. For undirected graphs, L is positive semidefinite; its kernel reduces to $x = (c\ c\ c ... c)^T$, $c \in \mathbb{R}$, if and only if G is connected. Thus H decreases unless all $Q_k\omega_k$ are equal. Orientations evolve in a compact set where V is bounded, and H is radially unbounded in the non-compact dimension corresponding to velocities. Therefore a LaSalle argument proves that the swarm converges to an invariant set where $Q_k\omega_k = Q_j\omega_j\ \forall j,k$, under the dynamics (1),(2) with $\tau_k = \tau_k^{(P)}$.

(b) Conservation of M is equivalent to $\sum_k Q_k \tau_k = 0$ which is easy to verify. For synchronization of the Q_k, the proof is in two steps. First, show that given a neighborhood $W \ni (Q_1 ... Q_N, \omega_1 ... \omega_N)$ of the set S_{M^*} of free rigid body motions with synchronized attitudes $Q_k(t) = Q_j(t)\ \forall j,k$ and total angular momentum $\|M^*\| < M_{max}$, there exist $|\sigma_1|$ and a neighborhood U of S_{M^*} such that starting in U implies staying in W if $|\sigma| > |\sigma_1|$. Then show that there exist $|\sigma_2|$ and a neighborhood W_1 of S_{M^*} such that for $|\sigma| > |\sigma_2|$, solutions of (1),(2) with identical $Q_k\omega_k$ and $\tau_k = \tau_k^{(P)}$ as in (4) that stay in W_1 are necessarily in S_{M_0}, where M_0 is the initial angular momentum of the system. Then taking $W = W_1$ and $|\sigma^*| > \max(|\sigma_1|, |\sigma_2|)$ concludes the proof.

For the first part, recalling $(d_{jk})^2 = 3 - \mathrm{trace}(Q_k^T Q_j)$, let $W = \{(Q_1 ... Q_N, \omega_1 ... \omega_N) : (d_{jk})^2 < \varepsilon\, \forall k,j$ and $\|M - M^*\| < \delta\}$. If E is the number of edges in G,

$$\tfrac{1}{2}\sum_k \sum_{j \rightsquigarrow k} (3 - \mathrm{trace}(Q_k^T Q_j)) = 3E - V(t)/\sigma < \varepsilon$$

(the factor $\frac{1}{2}$ comes from counting each distance twice) is sufficient for a solution starting with $\|M - M^*\| < \delta$ to be in W at time t. Since H decreases, for $t \geq 0$, $T(t) + V(t) \leq T(0) + V(0)$ so $V(t) - V(0) \leq T(0) - T(t) \leq T(0)$. Hence if $|\sigma| > |\sigma_1|$, then $(V(0) - V(t))/\sigma \leq T(0)/|\sigma_1|$ and so

$$3E - V(t)/\sigma \leq (3E - V(0)/\sigma) + T(0)/|\sigma_1|\ .$$

Choose a neighborhood $U_1 \subseteq W$ of S_{M^*} such that $\max_k \|Q_k J\omega_k - \frac{M^*}{N}\| < \frac{\beta}{N}\|M^*\|$, for some $\beta > 0$. Initial conditions in U_1 imply $T(0) < \frac{\|M^*\|^2}{2J_3 N}(1+\beta)^2$. Then taking (assuming actual M^* unknown) $|\sigma_1| > \frac{M_{max}^2}{\varepsilon J_3 N}(1+\beta)^2$ ensures $T(0)/|\sigma_1| < \frac{\varepsilon}{2}$. Also define U_2 such that $3E - V(0)/\sigma < \varepsilon/2$. Then with initial conditions in $U = U_1 \cap U_2$, the system stays in W for $t \geq 0$.

The second part involves more calculations, which will not all be detailed. Denote the final common velocity by $Q_k\omega_k = \Omega(t)\ \forall k$; note that $\|\Omega\| \leq \frac{\|M\|}{N J_3}$. The time derivative of $\Omega = Q_k\omega_k$ along solutions of the closed-loop system is (note that $\frac{d}{dt}(Q_k^T Q_j) = 0$ when $Q_k\omega_k = Q_j\omega_j$)

$$\frac{d}{dt}\Omega = -\sigma\, Q_k J^{-1} \sum_{l \rightsquigarrow k} [Q_k^T Q_l - Q_l^T Q_k]^\vee + Q_k J^{-1} Q_k^T [Q_k J Q_k^T \Omega]^\wedge\, \Omega \tag{7}$$

which must hold $\forall k$. Denoting the first and second terms of the right side of (7) by $(7a)_k$ and $(7b)_k$ respectively,

$$\|(7a)_k - (7a)_j\|^2 = \|(7b)_k - (7b)_j\|^2 \tag{8}$$

$\forall k,j$. The right side of (8) is bounded by (calculations)

$$\|(7b)_k - (7b)_j\|^2 \leq \tfrac{16 J_1^2}{J_3^2}\|\Omega\|^4 (d_{jk})^2\ .$$

Thus the same bound must hold for the left side of (8),

$$\|(7a)_k - (7a)_j\|^2 \leq \tfrac{16 J_1^2}{J_3^2}\|\Omega\|^4 (d_{jk})^2\ .$$

Summing the last condition over all k,j, using conservation of M and linearizing leads to (calculations)

$$\tfrac{2\sigma^2\lambda_2^3}{J_1^2}(d_{max}^2 + \mathcal{O}(d_{max}^4)) \leq \tfrac{16 J_1{}^2 E\, \|M^*\|^4}{J_3^6 N^2}\, d_{max}^2 + \mathcal{O}(d_{max}^3) \tag{9}$$

where $\lambda_2 > 0$ is the second-smallest eigenvalue of the Laplacian L of G and d_{max}^2 denotes the maximal value of $(d_{jk})^2$ among all pairs of connected agents. Choosing W_1 such that the higher-order terms represent less than $\gamma_1 < 1$ and $\gamma_2 < 1$ respectively on the left and right side of (9), the condition becomes

$$d_{max}^2 \leq \tfrac{8 J_1^4 E}{(1-\gamma_1)(1-\gamma_2) J_3^6 \lambda_2^3}\, \tfrac{\|M^*\|^4}{\sigma^2}\, d_{max}^2\ . \tag{10}$$

Taking $\sigma^2 > (\sigma_2)^2 := \frac{8 J_1^4 E\, M_{max}^4}{(1-\gamma_1)(1-\gamma_2) J_3^6 \lambda_2^3}$, (10) can only be satisfied if $d_{max}^2 = 0 \Leftrightarrow Q_k = Q_j\ \forall k,j$. □

[1] The Laplacian of a graph has entries $l_{kj} = -1$ if $j \rightsquigarrow k$ and $l_{kj} = 0$ otherwise for $j \neq k$; $l_{kk} = -\sum_{j \neq k} l_{kj}$.

Several comments are in order about Theorem 1.

- Theorem 1(a) still holds for time-varying (uniformly connected) and directed, but balanced graphs [2] because $x^T Lx$ is still non-negative in this case.
- When the swarm is synchronized, all control torques τ_k vanish. Hence, the (unimposed) motion of the synchronized swarm can be any free rigid body motion.
- Theorem 1(b) is a local result. However, simulations indicate a large basin of attraction U. The proof contains three conditions for U. Conditions with δ and β basically impose $\|M\|$ lower and upper bounded and $\|\omega_k\|$ upper bounded; for "infinite σ", this still allows almost any initial condition. The critical constraint is with ε chosen to bound high order terms in (9). For "infinite σ" relevant high-order terms are on the left; they are like $\sin^2(\theta) - \theta^2$ around $\theta = 0$. Then it is sufficient that all Q_k are inside a geodesic ball of radius $\pi/2$, also the maximal convex set of $SO(3)$. This is consistent with synchronization being the only minimum of V for all Q_k in a convex set.
- The bound on $|\sigma|$ reflects that the controller must overcome unknown "perturbations" $[J\omega_k]^\wedge \omega_k$. Sliding mode control like $\mathrm{grad_k}(V)/\|\mathrm{grad_k}(V)\|$ is similar to "infinite σ" near synchronization. However, the resulting chattering depends on controller parameters, and ω_k-dependent bounds will still be required to stay in a neighborhood of synchronization where the "perturbations" do not desynchronize the system. This even gets more difficult since $\frac{d}{dt}M \neq 0$ and $\frac{d}{dt}H \nleq 0$.
- With no condition on σ, there are situations arbitrarily close to synchronization but from which it is never reached. Take two agents A and B synchronized and rotating around e_3 with velocity Ω, where e_1, e_2, e_3 denote principal axes of $J_1 > J_2 > J_3$. Now (see Figure 1), with respect to this synchronized state, A and B are tilted by ϕ and $-\phi$ respectively around e_2, with ϕ arbitrarily small; they still rotate with $Q_A\omega_A = Q_B\omega_B = \Omega$ around the axis aligned with the initial synchronized e_3, which now makes an angle ϕ with actual axes e_{3A} and e_{3B}. Then $[J\omega_k]^\wedge \omega_k$ pulls A and B further apart, while $\tau_k^{(P)}$ pulls them together. For a particular ratio $\|\Omega\|^2/\sigma$, both effects exactly cancel.
- In [8], general forms for relative dissipation are proposed based on reduction techniques; they require that consecutive Poisson brackets of allowed torques restore full rank, which is not the case here.

4 Extension 2: directed and varying graphs

The previous algorithms require fixed, undirected graphs. "Consensus algorithms", see [34, 18, 21], *globally* synchronize variables in vector spaces *with directed and time-varying graphs*. Therefore, [26] embeds $SO(3)$ in $\mathbb{R}^{3\times3}$ and builds a consensus algorithm for auxiliary

[2] A directed graph is *balanced* when at each node, the number of incoming edges equals the number of outgoing edges.

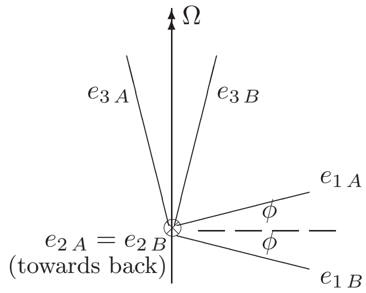

Fig. 1. Two rigid bodies in a situation from which (4),(6) do not synchronize attitudes for small σ. All vectors lie in the same plane, except $e_{2\,A} = e_{2\,B}$ perpendicular to the page.

variables $Y_k \in \mathbb{R}^{3\times3}$; Q_k tracks the projection of Y_k on $SO(3)$ such that synchronizing Y_k implies synchronizing Q_k. Here [26] is extended to mechanical system (1),(2).

The consensus algorithm is, in inertial frame,

$$\tfrac{d}{dt} Y_k = \beta \, \textstyle\sum_{j \rightsquigarrow k} (Y_j - Y_k) \, , \qquad \beta > 0 \qquad (11)$$

or equivalently in body frames with $X_k = Q_k^T Y_k$

$$\tfrac{d}{dt} X_k = \beta \, \textstyle\sum_{j \rightsquigarrow k} (Q_k^T Q_j X_j - X_k) \, - [\omega_k]^\wedge X_k \qquad (12)$$

for $k = 1...N$. Projection and tracking is implented with artificial potential

$$V = \sigma \textstyle\sum_k \mathrm{trace}(Q_k^T Y_k) \, , \qquad \sigma < 0 \, . \qquad (13)$$

Indeed, the distance from Y_k to $Q \in SO(3)$ in $\mathbb{R}^{3\times3}$ is

$$\begin{aligned} \|Y_k - Q\|^2 &= \mathrm{trace}((Y_k - Q)^T \, (Y_k - Q)) \\ &= 3 + \mathrm{trace}(Y_k^T Y_k) - 2 \, \mathrm{trace}(Q_k^T Y_k) \, . \end{aligned}$$

The resulting torque is

$$\tau_k = -\sigma \, [X_k - X_k^T]^\vee + \tau_k^{(D)} \, , \qquad k = 1...N \, . \qquad (14)$$

Controller (12),(14) only involves variables in body frame (autonomy) but, unlike extension 1, contains ω_k. Cost is added to store, update and exchange the X_k.

Theorem 2 *For G uniformly connected* [3] *, attitude synchronization with $\omega_k = 0 \, \forall k$ is almost globally asymptotically stable for (1),(2) with controller (12),(14),(5).*

PROOF. The Y_k evolve under (11) independently of the agents' motions. Hence classical consensus results (e.g. [18]) ensure their exponential convergence to a common constant Y_∞ for G uniformly connected. The Y_∞

[3] [18]: G is uniformly connected if there exist an agent k, and time periods T, $\Delta > 0$ such that, $\forall t$, there is a directed path from k to all other agents in the union of all edges appearing at least during Δ time units in the interval $[t, t + T]$.

such that $v(Q) := \sigma\,\mathrm{trace}(Q^T Y_\infty)$ has several minimizers on $SO(3)$, are non-generic [26]; the following assumes that $v(Q)$ has a unique minimizer $Q^* = \mathrm{Proj}_{SO(3)}(Y_\infty)$.

From the previous paragraph, (1),(2),(14),(5) form an asymptotically autonomous system where agents are decoupled; the limiting (autonomous) system is obtained by replacing X_k with $Q_k^T Y_\infty$. Solutions of an asymptotically autonomous system converge to a chain recurrent set of the limiting system [17]. A point x in state space is chain recurrent if and only if it belongs to the intersection of all locally asymptotically stable sets containing the positive limit set $L^+(x)$ of x [22]. The limiting system for k is of the "shaped energy" form with H bounded below and $\frac{d}{dt}H = -\gamma\,\|\omega_k\|^2$. A LaSalle argument on H as for Theorem 1(a) shows that the positive limit set $L^+(x)$ for the autonomous system of any point $x := (Q_k, \omega_k)$ only contains equilibria x_e, where $\omega_k = 0$ and Q_k is at a critical point of $v(Q_k)$; the set of equilibria is denoted by $\mathcal{E}$. Any $x \in \mathcal{E}$ is chain recurrent. For $x \notin \mathcal{E}$, define $\mu := \max_{y \in L^+(x)}(H(y)) < H(x)$ and $S := \{y : H(y) \leq \mu\}$. S contains $L^+(x)$ but not x, since $H(y) < H(x)$ for any $y \in L^+(x)$ when $x \notin \mathcal{E}$. For any Y_∞, $v(Q_k)$ takes a finite number of values at critical points [26]. Then there exists $\nu > \mu$ such that $H(y) \in (\mu, \nu) \Rightarrow y \notin \mathcal{E}$. Choose an ε-neighborhood S_ε of S and define $\rho = \min_{y \in (SO(3)\times\mathbb{R}^3)\setminus S_\varepsilon}(H(y)) > \mu$. Select a δ-neighborhood S_δ of S where $H(y) < \min(\{\nu, \rho\})$. Starting in S_δ ensures staying in S_ε so S is locally stable; every point of S_δ must converge to a point of $\mathcal{E} \cap S_\delta \subseteq \mathcal{E} \cap S$ so S is locally asymptotically stable. Thus S is an asymptotically stable set containing $L^+(x)$ but not x. Then from [22], non-equilibrium points are not chain recurrent: the autonomous system's chain recurrent set reduces to the critical points of $v(Q_k)$. For a generic Y_∞, the latter contain the unique minimum Q^* and three unstable points [26]. All solutions starting outside these three points and their stable manifolds converge to $Q_k = Q^*$. □

Theorem 2 deserves the following comments.

- Unstable solutions that do not converge to synchronization are (i) situations where $\mathrm{Proj}_{SO(3)}(Y_\infty)$ is not unique; (ii) a few unstable critical points of V in (13).
- The choice $X_k(0) = \alpha_k I_3$ with $\alpha_k \in \mathbb{R}^+\ \forall k$ avoids unnecessary transients when the $Q_k(0)$ are close.
- Embedding $SO(3)$ in $\mathbb{R}^{3\times3}$ uses 9-dimensional X_k. In fact, the 3-dimensional $SO(3)$ can be embedded in $\mathbb{R}^{3\times2}$ by only retaining the first two columns of Q_k. It is even possible to embed $SO(3)$ in $\mathbb{R}^5$ [33].
- The proof of Theorem 2 uses vector space consensus in cascade with projection and tracking for the actual system. [26] applies this to first-order integrators. [27] consider (1),(2) with "consensus tracking"; "energy shaping" control is expected to be more robust.
- Dissipation (5) forces 0 final velocities. It is tempting to combine extensions 1 and 2 to obtain global convergence without restricting final motion. Unfortunately this is not easy, independently of $\tau_k^{(D)}$, because the final motion is dictated by $Q^* = \mathrm{Proj}(Y_\infty)$ and (11) leads to a *constant* Y_∞. Different adaptations have been explored in simulation, with no conclusive result.

5 Conclusion

This paper presents two extensions of results in [31, 19] for rigid body attitude synchronization with limited communication links and no external reference.

A first controller avoids imposing the final motion of the swarm. For fixed, undirected, connected communication graphs, angular velocities globally synchronize. Asymptotic attitude synchronization is local and requires adapting the strength of the interaction potential to the initial total angular momentum. These limitations illustrate difficulties, in accordance with insights of formal reduction techniques [6], encountered when using controllers with configuration and velocity symmetry for mechanical systems that just have configuration symmetry.

A second controller achieves almost-global attitude synchronization for time-varying and directed graphs, at the cost of introducing auxiliary variables that communicating agents exchange; connectedness can be relaxed to the union of all links appearing in a fixed time span.

Combining both extensions remains speculative. Future work could add actuator constraints to the communication constraints. Also, similar proofs can likely be repeated for mechanical systems on other Lie groups.

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