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SYNCHRONIZATION WITH PARTIAL STATE COUPLING ON $SO(n)$*

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Abstract. This paper studies autonomous synchronization of k agent orientations $\in SO(n)$ by using links that each convey (a priori different) partial state information between two agents: the latter compare the action of their states on a common link-dependent vector of $\mathbb{R}^n$. Such setting appears when each pair of agents compares the expression, in their respective body frames, of a common external direction (force field, heading of moving object, etc.). To synchronize their states, agents must combine this information over time or throughout the network. A gradient coupling law for synchronization is proposed. Extensive convergence analysis of the coupled agents is provided, both for fixed and time-varying reference vectors. The results are compared to a similar setting with states in $\mathbb{R}^n$ and agents coupled through scalar product of their state with a reference vector.

Key words. consensus, attitude synchronization, output coupling, special orthogonal group

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1. Introduction.

Synchronization and collective phenomena have recently drawn considerable attention. Inspired by natural systems (see, e.g., [32, 37, 41]), the control community has developed an interest in self-coordination of multiagent systems; applications include communication networks and vehicle formations (see, e.g., [10, 13, 14, 15, 16, 22, 25, 27, 38]). The literature addresses aspects like optimal configuration, collision avoidance, nonlinear dynamics, interaction graph structure, and distributed controller tuning. A distinction must be made between decentralized or distributed controllers that let a set of coupled subsystems follow (sometimes implicit) reference inputs are designed for a set of coupled subsystems [6, 42], and *autonomous coordination*, where systems evolve without external command and a collective behavior emerges from their coupling only. Our work belongs to the latter framework.

A basic autonomous collective behavior, called "synchronization" or "consensus," is that agents reach a common state value thanks to pairwise interactions. This problem has been studied mainly with a standard linear algorithm for states in $\mathbb{R}^n$ (see e.g., [15, 28, 38]; [26] proposes a more general convexity argument). Motivated by problems involving data on the sphere, agents representing subspaces, or rigid body attitudes, [34] extends consensus to nonlinear manifolds. Synchronization of satellite attitudes, evolving on the space $SO(3)$ of rotation matrices, has been studied both in the presence of a reference or leader [7, 18, 21, 40] and in fully autonomous (i.e., leader-

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and referenceless) framework [27, 31, 35], mostly with second-order mechanical dynamics. All these papers consider *full state exchange*[1] between interacting agents.

In this context, the present paper introduces two innovations.

First, we initiate a study of synchronization with *partial state coupling on the Lie group $SO(n)$*: a basic state coupling law for synchronization on $SO(n)$ (see [34]) is extended to a setting where information coupling the agents contains only the action of their state on a "reference vector" in $\mathbb{R}^n$. This output map is inspired by a proposal in [29] for agents on the circle, that is, $SO(2)$; note, however, that on $SO(2)$ this output is equivalent to full state knowledge—the concern of [29] is noise reduction. Such output maps are also considered for observers on $SO(3)$ (see, e.g., [8, 24]); they appear when an external vector field (gravity, magnetic field) is measured in an agent-fixed frame. Observer or tracking algorithms can be viewed as a special case of synchronization, between two agents only (the actual state and the estimated or to-track state), involving a single link. We here extend this formalism to synchronizing k agents, giving the result for $k = 2$ as a first approach that admits a complete analysis: a single fixed reference vector is not sufficient to synchronize the states, but with a persistently varying reference vector the states can synchronize as the "averaged output map" gives access to the full state.

Second, for $k > 2$ agents, we propose an *alternative setting for partial state coupling in (any) network*. Consensus under partial state coupling has been studied in, e.g., [36, 39, 23] for linear systems. There each agent is viewed as a copy of the same linear system, whose single output is compared with others. A controller synchronizing outputs achieves full state agreement thanks to detectable zero-input agent dynamics. In contrast, the present paper considers that one agent can involve different outputs in its interaction with different neighbors. *This setting views "outputs" as an interaction-defined characteristic rather than an agent-centered characteristic.* We then follow the initial consensus framework, with no zero-input dynamics, and study how full state synchronization can be recovered by exchanges of different state parts through the whole network. Note that in the setting of [36, 39, 23], having no zero-input dynamics makes the individual systems undetectable and state consensus would never be reached. We therefore also briefly give results corresponding to our particular partial state coupling framework for states evolving on $\mathbb{R}^n$. Unlike what first intuition might suggest, each agent sharing, through its different outputs, information about its whole state with the network, is not sufficient for synchronization.

Beyond settings where it results from physical constraints (see section 2), partial state coupling could also be an explicit design choice in an attempt to reduce information transfer for, e.g., satellite attitudes synchronization.

The paper is organized as follows. Section 2 motivates and formalizes the setting, proposes a cost function, and derives a gradient law on $SO(n)$ that couples the agents via available information. The analog problem on $\mathbb{R}^n$ is also introduced. Section 3 starts the convergence analysis with two agents. Section 4 presents a detailed analysis for networks of $k > 2$ agents. Section 4.2 gives the result for time-varying reference vectors, adding a technical assumption with respect to (w.r.t.) the two-agent case. Section 4.1 characterizes local state synchronization for fixed reference vectors. We give (i) a necessary condition in terms of individual agent properties, (ii) a necessary and sufficient condition on $\mathbb{R}^n$ and sufficient condition on $SO(n)$ in terms of the rank of a generalized Laplacian matrix including the effect of reference vectors, and (iii) a tighter sufficient condition on $SO(n)$ which involves a matrix of size kn^2 for k agents.

[1] Or at least full configuration exchange for second-order systems.

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For generic y_{ij}, these translate into conditions on the network structure. Several (counter-)examples are given. Section 5 studies $SO(3)$ with fixed reference vectors in more detail, deriving sufficient conditions for which output synchronization and state synchronization coincide; this property is stronger than local stability of state synchronization under the gradient flow. We also draw attention to poor robustness of a setting where reference vectors correspond to relative positions of the agents in $\mathbb{R}^3$. Simulations are provided for illustration.

1.1. Notation. $I_n \in \mathbb{R}^{n\times n}$ is the identity matrix. X^T is the transpose of $X \in \mathbb{R}^{n\times m}$ and $\mathrm{skew}(X) = \frac{1}{2}(X - X^T)$ for $m = n$. The Euclidean norm of column vector $y \in \mathbb{R}^n$ is denoted $\|y\|_2 := \sqrt{y^T y}$; the Frobenius norm of $M \in \mathbb{R}^{n\times n}$ with entries m_{ij} is $\|M\|_F := \sqrt{\mathrm{tr}(M^T M)} = \sqrt{\sum_{ij} m_{ij}^2}$. For $X \in \mathbb{R}^{n\times m}$ with column vectors $x_1, x_2, \ldots, x_m \in \mathbb{R}^n$, we write $\mathrm{rank}(X) = \mathrm{rank}(x_1, x_2, \ldots, x_m)$ its rank and $\mathrm{vec}(X) \in \mathbb{R}^{nm}$ the vector obtained by stacking $x_1, x_2, \ldots, x_m$ in a single column. For a set $S \subset (\mathbb{R}^{n\times n})^k$ we denote $\mathrm{vec}(S) = \{\mathrm{vec}((X_1^T, \ldots, X_k^T)^T) : (X_1, \ldots, X_k) \in S\} \subset \mathbb{R}^{kn^2}$. The vector product of $x_1, x_2 \in \mathbb{R}^3$ is $x_1 \times x_2 \in \mathbb{R}^3$. The Kronecker product of matrices is denoted $\otimes$. The unit sphere in $\mathbb{R}^n$ is denoted $S^{n-1} = \{x \in \mathbb{R}^n : \|x\|_2 = 1\}$.

$SO(n) \cong \{Q \in \mathbb{R}^{n\times n} : Q^T Q = I_n, \det(Q) = 1\}$ is the Lie group of n-dimensional rotations; $\mathfrak{so}(n) \cong \{X \in \mathbb{R}^{n\times n} : X^T = -X\}$ is the Lie algebra of $SO(n)$, that is, the tangent space to $SO(n)$ at identity. We consider $SO(n)$ equipped with the standard bi-invariant Riemannian metric induced by its embedding in $\mathbb{R}^{n\times n}$: $\langle Q\Omega, Q\Theta\rangle = \mathrm{tr}(\Omega^T\Theta)$, where $\{Q\Omega, Q\Theta\} \subset T_Q SO(n)$ the tangent space to $SO(n)$ at Q, and $\{\Omega, \Theta\} \subset \mathfrak{so}(n)$. $SO(n)$ acts on $\mathbb{R}^n$ by matrix vector multiplication $(Q, y) \mapsto Qy$. The stabilizer of $y \in \mathbb{R}^n$ with respect to this action is $\mathrm{stab}(y) = \{Q \in SO(n) : Qy = y\}$. For $y \neq 0$ this stabilizer is a subgroup homomorphic to $SO(n-1)$; in particular, $Q \in \mathrm{stab}\, y$ if and only if $Q^{-1} = Q^T \in \mathrm{stab}\, y$.

An undirected (finite) graph $\mathcal{G}(\mathcal{V}, \mathcal{E})$ consists of $\mathcal{V}$ a finite set of vertices—in the following representing the agents—and $\mathcal{E}$ a set of unordered vertex pairs called edges—representing undirected interaction links among agents. We label vertices (i.e., agents) by integers: $\mathcal{V} = \{1, \ldots, k\}$. We denote by $\#\mathcal{E}$ the cardinality of $\mathcal{E}$. Following a customary abuse of notation we write $(i, j) \in \mathcal{E}$ to actually represent the unordered pair $\{i, j\} \in \mathcal{E}$. Vertices i, j linked by edge $e = (i, j) \in \mathcal{E}$ are denoted $i = vl(e) \in \mathcal{V}$ and $j = vr(e) \in \mathcal{V}$ with $vl(e) < vr(e)$. $\mathcal{G}(\mathcal{V}, \mathcal{E})$ can be represented by its adjacency matrix $A = (a_{ij})$, where $a_{ij} = 1$ if $(i, j) \in \mathcal{E}$ and $a_{ij} = 0$ otherwise. In particular, $\mathcal{G}$ undirected implies $a_{ij} = a_{ji}$ for all i, j, i.e., A is symmetric; we consider $\mathcal{G}$ without self-loops (edges between a vertex and itself), so $a_{ii} = 0$. In the *incidence matrix* $B \in \mathbb{R}^{k\times\#\mathcal{E}}$, each column corresponds to an edge e of $\mathcal{G}$ and contains all zeros except 1 on row $vr(e)$ and -1 on row $vl(e)$. For undirected graphs, the standard Laplacian matrix is $L = BB^T = D - A$, where diagonal matrix D contains $d_{ii} = \sum_{j=1}^k a_{ij}$ the *degree* of vertex i, i.e., number of edges involving i.

2. Problem statement and coupling law.

2.1. Partial state coupling on $SO(n)$. Consider a set of k rigid body "agents"—e.g., robots or satellites; each rigid body has its own reference ("body") frame orientation $Q_i \in SO(n)$, $n \geq 2$, for $i = 1, 2, \ldots, k$. A basic functionality for autonomous collective task solving is for the agents to autonomously reach agreement on a common orientation $Q_1 = Q_2 = \cdots = Q_k$, i.e., to *synchronize* orientations. Note that in the agreement problem setting one might consider—in a first instance, depending on applications—adapting the reference frames Q_i without actually physically moving

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the rigid bodies. We consider synchronization at this planning level, and therefore assume that orientations are fully actuated and velocity controlled, according to left-invariant dynamics,

$$\frac{d}{dt}Q_i = Q_i\, u_i, \quad i = 1, 2, \ldots, k\,. \tag{1}$$

The control inputs u_i are expressed in respective body frames, e.g., u_i for $n = 3$ contains the components of angular velocity expressed in the body frame of agent i. The goal in the present paper is to reach *state synchronization* by just appropriately *coupling* agents through u_i.

DEFINITION 1. *State synchronization set C_s is the set of states such that all agents have the same state, i.e., $C_s := \{(Q_1, Q_2, \ldots, Q_k) \in (SO(n))^k : Q_1 = Q_2 = \cdots = Q_k\}$.*

As coupling every agent to every other agent could lead to prohibitive communication cost for large k, it is customary in the consensus literature to consider an imposed restricted network coupling constraint:

(R1) *Network coupling:* admissible pairwise interactions are represented by an undirected graph $\mathcal{G}(\mathcal{V}, \mathcal{E})$ such that u_i only uses information from agents j for which $(i, j) \in \mathcal{E}$.

Traditional approaches to orientation synchronization like [27, 31, 33, 34] assume that connected agents $(i, j) \in \mathcal{E}$ have access to their relative orientation $Q_i^T Q_j$. More generally, most synchronization studies assume relative *state* measurements. However, in practice it is not obvious how to directly measure the relative orientation between two rigid bodies in real time. In contrast, measuring the expression of a vector (direction) in a body frame is common. One could imagine sensors mounted on the agents that are sensitive to magnetic, electric, or gravitational fields, or vision-based detectors observing the position in each body frame of some striking object (light source, beacon, other robot). A vector y in (hypothetical) inertial frame is measured as $Q_i^T y$ in the body frame of agent i. Observers based on such output—comparing $Q^T y$ to y—are proposed in, e.g., [8, 24].

Without relying on an external reference frame, two agents i and j can still compare how they see, in their respective body frames, a common given vector which has unknown description $y_{ij} \in \mathbb{R}^n$ in an inertial frame. That is, agents i and j respectively measure $Q_i^T y_{ij}$ and $Q_j^T y_{ij}$; then both agents communicate to compare coordinates

$$Q_i^T y_{ij} \text{ and } Q_j^T y_{ij}. \tag{2}$$

We therefore consider the following coupling possibility:

(R2) *Partial state coupling:* interacting agents i and j exchange outputs (2) with reference vectors $y_{ij} = y_{ji} \in S^{n-1}$ given but a priori unknown. *Importantly, we allow y_{ij} to differ from link to link.*

The last part is in strong contrast with existing studies where "output maps" are associated to agents (nodes of the graph) rather than to links (edges of the graph); see, e.g., [36, 39, 23]. Restriction to unit vectors is not essential, as vector magnitudes are independent of reference frame orientation. Magnitudes might therefore in fact serve as a way for a pair of agents i, j to identify a relevant common vector y_{ij} in their environment. This could be a sufficiently homogeneous magnetic, electric, or gravitational field. Or, both agents may see the same object moving in their neighborhood, so that its velocity vector may serve as y_{ij}; one might indeed encounter the situation that each agent pair is able to identify a common such object, but no single object is visible by the whole swarm. Another possible y_{ij} is the position of some object, like

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a reference star, that is far enough w.r.t. interagent distances such that the parallax effect due to different positions of the two agents is negligible. Finally, a more special situation would derive y_{ij} from positions y_i, y_j of the rigid bodies i, j in $\mathbb{R}^n$: agent i measures in its frame the relative position $Q_i^T(y_j - y_i)$ of j, and agent j similarly measures $Q_j^T(y_i - y_j)$; these are equivalent to (2) by (i) taking $y_{ij} = \frac{y_j - y_i}{\|y_j - y_i\|_2}$ and (ii) introducing a minus sign in the comparison of the two body frame measurements that reflects $(y_i - y_j) = -(y_j - y_i)$. This last situation may seem attractive to implement in confined environments, but if all y_{ij} are built in this way, then they have a particular interdependence that can be problematic; see section 5.

DEFINITION 2. *Output synchronization set C_o is the set of states such that both outputs are equal on each link, i.e., $C_o := \{(Q_1, Q_2, \ldots, Q_k) \in (SO(n))^k : Q_i^T y_{ij} = Q_j^T y_{ij}$ for all $(i,j) \in \mathcal{E}\}$.*

Clearly, $C_s \subseteq C_o$. The main question, investigated in this paper, is under which conditions output synchronization implies state synchronization. Both fixed and time-varying y_{ij} are considered. The challenge starts for $n > 2$: for $n = 2$, as studied in [29], $SO(2)$ is isomorphic to the circle and (2) is equivalent to full state information.

Taking state synchronization on $SO(n)$ from state-based to output-based coupling gives it a more fundamental scope. If two agents i, j know their relative state $Q_i^T Q_j$, then they can "talk the same language," in the sense that coordinates $Q_i^T v$ of a vector expressed in the body frame of i can be readily "translated" into coordinates $Q_j^T v = (Q_i^T Q_j)^T Q_i^T v$ of the same vector in the body frame of j. In contrast, when $Q_i^T Q_j$ is unknown, agent i cannot send a vector message (e.g., desired velocity) to agent j in an appropriate coordinate expression. Therefore, in the presence of output measurements (2), synchronizing the $Q_i \in SO(n)$ can be viewed as in fact providing a means to communicate through "coordinate" messages.

In the following we design the u_i as a static function of (2). In our fully autonomous setting, this is no limitation w.r.t. local-observer-based strategies as in, e.g., [23] for the following reason. To avoid relying on convergence to an external reference, an isolated agent must follow $\frac{d}{dt} Q_i = Q_i\, \Omega(t)$ with *state-independent* rotation rate expressed in the body frame, $\Omega(t)$. This allows no stabilizing zero-input dynamics. Moreover, adding the same velocity $\Omega(t)$ to the input of all agents in (2) does not modify $\frac{d}{dt}(Q_j Q_i^T\, y_{ij} - y_{ij})$, so we can as well take $\Omega(t) = 0$: there is no possibility to exploit local agent dynamics for retrieving state consensus as in, e.g., [36, 39, 23]. For this reason, recovering full state consensus requires each agent to share several pieces of information along its links.

2.2. Gradient flow on $SO(n)$. A gradient flow for (at least local) synchronization on $SO(n)$ with *full state coupling* is proposed in [34]. It expresses state disagreement by cost function

$$f_s = \frac{1}{2} \sum_{(i,j)\in\mathcal{E}} \|Q_i - Q_j\|_F^2 = \sum_{(i,j)\in\mathcal{E}} (n - \operatorname{tr}(Q_i^T Q_j)) \tag{3}$$

and makes agents "rotate toward each other" according to

$$\frac{d}{dt} Q_i = -\operatorname{grad}_{Q_i}(f_s) = Q_i \sum_{\{j:(i,j)\in\mathcal{E}\}} \operatorname{skew}(Q_i^T Q_j), \quad i = 1, 2, \ldots, k\,. \tag{4}$$

The gradient is computed w.r.t. the product metric on $SO(n)^k$ of the bi-invariant Riemannian metric for $SO(n)$ embedded in $\mathbb{R}^{n\times n}$; see section 1.1. Algorithm (4) requires agents to compare relative states, $Q_i^T Q_j$. For agents only comparing outputs

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(2), we propose a gradient flow which minimizes the output disagreement cost function

$$f_o = \frac{1}{2} \sum_{(i,j)\in\mathcal{E}} \|Q_i^T y_{ij} - Q_j^T y_{ij}\|_2^2 = \sum_{(i,j)\in\mathcal{E}} (1 - \mathrm{tr}(Q_i^T M_{ij} Q_j)), \tag{5}$$

where $M_{ij} := y_{ij} y_{ij}^T$ is a rank 1 projector. This yields a coupling law satisfying R1 and R2:

$$\frac{d}{dt} Q_i = -\,\mathrm{grad}_{Q_i}(f_o) = \; Q_i \sum_{\{j:(i,j)\in\mathcal{E}\}} \mathrm{skew}(Q_i^T M_{ij} Q_j), \quad i = 1, 2, \ldots, k\,. \tag{6}$$

Both dynamics (4) and (6) are right-invariant w.r.t. absolute orientation: if $(Q_1(t), \ldots, Q_k(t))$ is a solution, then $(Q_1(t)R, \ldots, Q_k(t)R)$ is also a solution for any constant $R \in SO(n)$.

Remark 1. We consider first-order dynamics for simplicity. Gradient laws can be extended to higher-order systems, e.g., applying forces or torques that are the gradient of a potential in second-order mechanical systems; see, e.g., [33] for extending (4) to mechanical systems with Euler equation dynamics.

Remark 2. Dynamics (6) can also be mapped to the space of outputs and viewed as a gradient in the submanifold of $M := (\,S^{n-1}\,)^{2\#\mathcal{E}}$ given by $N := \{(r_{e_1}, \ldots, r_{e_{\#\mathcal{E}}}, s_{e_1}, \ldots, s_{e_{\#\mathcal{E}}})\colon r_e = Q_{vl(e)}^T y_e$ and $s_e = Q_{vr(e)}^T y_e$ for some $(Q_1, Q_2, \ldots, Q_k) \in (SO(n))^k$, for all $e \in \mathcal{E}\}$. The corresponding dynamics on N extend to gradient dynamics of $h_o = \sum_{e\in\mathcal{E}} \|r_e - s_e\|_2^2$ on M for a suitable Riemannian metric. Explicitly, (6) on the output space translates to

$$\begin{aligned} \frac{d}{dt} r_e &= -\sum_{j\in \mathrm{ed}(vr(e))} \mathrm{skew}(r_j\, s_j^T)\, r_e \quad \text{and} \\ \frac{d}{dt} s_e &= -\sum_{j\in \mathrm{ed}(vl(e))} \mathrm{skew}(s_j\, r_j^T)\, s_e \quad \text{for } e \in \mathcal{E}\,, \end{aligned} \tag{7}$$

where $\mathrm{ed}(i)$ denotes the set of edges involving vertex i. The appearance of $r_j\, s_j^T$, with $j \neq e$, in $\frac{d}{dt} r_e$ expresses interdependence of all outputs r_j, s_j that correspond to the same state Q_i. This interdependence must be encoded in a nontrivial geometry to make dynamics (7) a gradient flow on N. In this interpretation, output synchronization corresponds to the set $N \bigcap \{(r_{e_1}, \ldots, r_{e_{\#\mathcal{E}}}, r_{e_1}, \ldots, r_{e_{\#\mathcal{E}}}) : r_{e_j} \in S^{n-1}\}$. Examining this intersection seems to be a challenging geometric problem.

2.3. Gradient flow with partial state coupling on $\mathbb{R}^n$. We compare our results to an analog linear setting. Consider k agents with states $x_i \in \mathbb{R}^n$ for $i = 1, 2, \ldots, k$, e.g., representing positions of agents in $\mathbb{R}^n$, $n \geq 1$. They evolve according to $\frac{d}{dt} x_i = u_i$, where $u_i \in \mathbb{R}^n$ must be designed as an interagent coupling to reach state synchronization $x_i = x_j$ for all i, j. The coupling is subject to restrictions (R1) and (R2): the network constraint remains unchanged, while partial state coupling is imposed by comparing, for each edge (i, j), the projection of x_i and x_j on some direction $y_{ij} \in S^{n-1} \subset \mathbb{R}^n$, i.e., communicated outputs are $x_i^T y_{ij}$ and $x_j^T y_{ij}$. So information exchanged along one edge is a scalar. An extension to projections onto subspaces, with $Y_{ij} \in \mathbb{R}^{n\times m}$, could also be considered. The natural cost function for this setting is

$$g_o \;=\; \frac{1}{2} \sum_{(i,j)\in\mathcal{E}} (x_i^T y_{ij} - x_j^T y_{ij})^2 \;=\; \frac{1}{2} \sum_{(i,j)\in\mathcal{E}} (x_i - x_j)^T\, M_{ij}\, (x_i - x_j)$$

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with $M_{ij} = y_{ij} y_{ij}^T$ defined as in section 2.2. The ensuing gradient flow is

$$\frac{d}{dt} x_i = \sum_{\{j:(i,j)\in\mathcal{E}\}} M_{ij}\,(x_j - x_i), \quad i = 1, 2, \ldots, k\,. \tag{8}$$

One term of (8) drives the components along direction y_{ij} of x_i and x_j toward each other, leaving the rest of the state unchanged. The idea is that by combining information along different directions y_{ij} of different edges, synchronization is reached on the whole state vector. This setting essentially differs from [39, 36, 23], where y_{ij} is the same for all edges and state synchronization relies on zero-input dynamics $\frac{d}{dt}x_i = Ax_i$ making an individual system detectable from its single output.

3. Convergence for two agents. For $k = 2$ agents, there is a single edge $(1, 2)$ with reference vector y_{12}. Moreover, invariance w.r.t. absolute orientation (or position) allows us to reduce the state of the system to the relative state of one agent with respect to the other one. Convergence properties on $SO(n)$ and on $\mathbb{R}^n$ are similar: for fixed y_{12}, synchronization appears only along the space on which information is projected; with varying y_{12} full state synchronization is possible.

On $SO(n)$, defining $Q = Q_1 Q_2^T$ leads to reduced dynamics,

$$\text{full state:} \quad \frac{d}{dt} Q = 2Q\,\mathrm{skew}(Q^T), \tag{9}$$

$$\text{partial state:} \quad \frac{d}{dt} Q = 2Q\,\mathrm{skew}(Q^T M_{12})\,. \tag{10}$$

These expressions are also valid for a directed link, e.g., when Q_1 follows (6) and Q_2 remains static. Dynamics (9) and (10) are also building blocks for tracking or observer algorithms: if an external pilot imposes $\frac{d}{dt}Q_2(t) = Q_2\,\Omega(t)$ with known input $\Omega(t) \in \mathfrak{so}(n)$, then (10) remains valid if $\frac{d}{dt}Q_1 = Q_1\,(\mathrm{skew}(Q_1^T M_{12} Q_2) + \Omega(t))$. Geometric observers resembling (10) are built on this basis; see, e.g., [8, 20, 24].

Dynamics (9) and (10) implement gradient descent for respective cost functions $\hat{f}_s(Q) = \|Q^T - I_n\|_F^2$ and $\hat{f}_o(Q) = \|Q^T y_{12} - y_{12}\|_2^2$. In terms of Q, state and output synchronization map to $\hat{C}_s = \{I_n\}$ and $\hat{C}_o = \{Q : \hat{f}_o(Q) = 0\} = \mathrm{stab}(y_{12})$, respectively. For (9), $Q = I_n$ is the only stable equilibrium; see [34]. For (10) the system behaves as follows.

THEOREM 1. *Consider the dynamics* (10) *on* $SO(n)$ *with* $n > 2$.

(a) *Let* $Z_y = \{Q : Qy_{12} = -y_{12}\}$ *be the set of maxima of* $\hat{f}_o$. *If* y_{12} *is constant, then all trajectories with initial condition* $Q(0) \in SO(n) \setminus Z_y$ *asymptotically converge to* $\hat{C}_o$; *convergence is locally exponential. Only initial points in* $S_0 = \{I_n + (\cos\theta - 1)\,(y_{12}y_{12}^T + zz^T) + \sin\theta\,(y_{12}z^T - zy_{12}^T)$ *with* $z \in S^{n-1}$, $z^T y_{12} = 0$, $\theta \in (-\pi, \pi)\}$, *converge to* $\hat{C}_s$.

(b) *If* $y_{12}(t)$ *is a time-varying smooth function,* $\frac{d}{dt}y_{12}$ *is uniformly bounded, and there exist* $T, \alpha > 0$ *such that* $B(t) := \frac{1}{T}\int_0^T M_{12}(t+\tau)\,d\tau$ *has at least* $n-1$ *eigenvalues larger than* α *for all* $t > 0$, *then* $\hat{C}_s$ *is the only asymptotically stable limit set and attracts (at least) all solutions starting with* $\hat{f}_s(Q) < 2$.

Proof. (a) Take $\hat{f}_o$ as Lyapunov function. Along solutions of (10), $\frac{d}{dt}\hat{f}_o = -\|\frac{d}{dt}Q\|_F^2 = -(1 + y_{12}^T Q y_{12})\hat{f}_o \le 0$. $SO(n)$ is compact and all functions are smooth, so by the LaSalle invariance principle (see, e.g., [17]) all solutions converge to the set where $\frac{d}{dt}\hat{f}_o = 0$. Points with $1 + y_{12}^T Q y_{12} = 0$ belong to Z_y; since they are maxima of $\hat{f}_o$, they attract no trajectories. So all trajectories starting outside Z_y converge to

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$\hat{f}_o = 0$. In the neighborhood of $\mathrm{stab}(y_{12})$, we have $y_{12}^T Q y_{12} > 0$, thus $\frac{d}{dt}\hat{f}_o < -\hat{f}_o$, which yields exponential convergence.

A trajectory $Q(t)$ of (10) starting at $Q(0) \in S_0$ stays in S_0 for all $t > 0$; writing Q as in the definition of S_0, dynamics reduce to $\frac{d}{dt}z = 0$, $\frac{d}{dt}\theta = -\frac{1}{2}\sin(\theta)$. Thus θ converges to 0 for $\theta(0) \neq \pi$, so points in S_0 converge to $\hat{C}_s = \{I_n\}$. Conversely, denote $\bar{S}_0 = \{I_n + (\cos(\theta)\text{-}1)\,(y_{12}y_{12}^T + zz^T) + \sin(\theta)\,(y_{12}z^T - zy_{12}^T) \,:\, z \in S^{n-1},\, z^T y_{12} = 0,\, \theta \in [-\pi, \pi]\}$. Note that $\bar{S}_0 \setminus S_0 \subset Z_y$. Any $Q \in SO(n)$ can be written $Q = RU$ with $R \in \mathrm{stab}(y_{12})$ and $U \in \bar{S}_0$; this is checked, e.g., by expressing Q in an orthonormal basis with first element y_{12} and showing that there is at least one solution (R, U) to $Q = RU$ for any $Q \in SO(n)$. Now if $U^*(t)$ is a solution of (10), then $RU^*(t)$ is also a solution of (10), provided $R \in \mathrm{stab}(y_{12})$. Therefore solutions starting at $Q(0) = RU(0)$ with $U(0) \in S_0$ converge to R; with $U(0) \in \bar{S}_0 \setminus S_0 \subset Z_y$ keep constant $\hat{f}_o$. Altogether solutions starting outside S_0 do not converge to $\hat{C}_s = \{I_n\}$.

(b) Take $\hat{f}_s$ as Lyapunov function: $\frac{d}{dt}\hat{f}_s = -2\,\mathrm{tr}(\frac{dQ}{dt}) = 2((Qy_{12})^T(Q^T y_{12}) - 1) \leq 0$ along solutions of (10), with equality at time t if and only if $Q^2(t) \in \mathrm{stab}(y_{12}(t))$. Points outside $\{I_n\}$ cannot be stable since I_n is the only local minimum of $\hat{f}_s$, which never increases along trajectories.

Since $y_{12}(t)$ is time-varying, a modified version of the LaSalle invariance principle must be used [2]. For a given set S_1, it ensures that all $Q(0) \in SO(n) \setminus S_1$ asymptotically converge to $\{I_n\}$ if, in addition to standard LaSalle conditions and technical requirements that automatically hold thanks to compactness of $SO(n)$ and smoothness of (10), the following hold: (i) solutions starting in $SO(n) \setminus S_1$ at $t = 0$ remain in $SO(n) \setminus S_1$ for all $t > 0$; (ii) denoting by $Q(t + \sigma; \bar{Q}, t)$ the solution at time $t + \sigma$ of (10) with initial condition $Q(t) = \bar{Q}$, there exists for any $\bar{Q} \in SO(n) \setminus (S_1 \cup \{I_n\})$ a finite time $\tau(\bar{Q})$ such that $\limsup_{t\to\infty} \hat{f}_s(Q(t + \tau(\bar{Q}); \bar{Q}, t)) < \hat{f}_s(\bar{Q})$.

Taking $S_1 = \{Q : \hat{f}_s(Q) \geq 2\}$, (i) holds since $\frac{d}{dt}\hat{f}_s \leq 0$ along trajectories. We now show that (ii) holds for $\tau(\bar{Q}) = T$. Assume that this is not the case, i.e., there is a $\bar{Q} \in SO(n) \setminus S_1$ such that

$$\limsup_{t\to\infty} \hat{f}_s(Q(t + \sigma; \bar{Q}, t)) = \hat{f}_s(\bar{Q}) \quad \text{for all } \sigma \in [0, T]. \tag{11}$$

Since $\frac{d}{dt}y_{12}$ is uniformly bounded, $y_{12}(t)$ is uniformly continuous. Therefore $\frac{d}{d\sigma}\hat{f}_s(Q(t+\sigma; \bar{Q}, t))$ is uniformly continuous (in σ and t) and it is necessary for satisfying (11) that

$$\begin{aligned} 0 &= \limsup_{t\to\infty} \min_{\sigma\in[0,T]} \tfrac{d}{d\sigma}\hat{f}_s(Q(t + \sigma; \bar{Q}, t)) \\ &= \limsup_{t\to\infty} \min_{\sigma\in[0,T]} 2\left(y_{12}(t+\sigma)^T\,(Q(t+\sigma; \bar{Q}, t))^2\, y_{12}(t+\sigma)\ - 1\right). \end{aligned} \tag{12}$$

This requires the existence, for any given $\varepsilon > 0$, of a sequence $t_m \to \infty$ for which $1 - y_{12}(t_m + \sigma)^T (Q(t_m + \sigma; \bar{Q}, t_m)^T)^2 y_{12}(t_m + \sigma) < \varepsilon$ for all $\sigma \in [0, T]$. Using the Jordan normal form of Q, note that all $Q \in SO(n)$ with eigenvalues $e^{i\phi_j}$, $|\phi_j| < \pi/2$, satisfy $y^T Q^2 y \leq y^T Q y$ for all $y \in S^{n-1}$. This is interesting since, as $\bar{Q} \notin S_1$, there exists $\beta(\bar{Q}) > 0$ such that $\hat{f}_s(Q) = 2(n - \mathrm{tr}(Q)) = 2(n - \sum_{j=1}^n \mathrm{real}(e^{i\phi_j})) \leq 2 - 2\beta$, equivalently $\sum_{j=1}^n \cos(\phi_j) \geq n - 1 + \beta$; which necessarily implies $|\phi_j| \leq \pi/2 - \beta$ for all j. Therefore,

$$\begin{aligned} \varepsilon &> y_{12}(t_m + \sigma)^T (I_n - Q^2(t_m + \sigma; \bar{Q}, t_m)) y_{12}(t_m + \sigma) \\ &\geq 1 - (y_{12}(t_m + \sigma)^T Q(t_m + \sigma; \bar{Q}, t_m) y_{12}(t_m + \sigma)) \end{aligned}$$

for all $\sigma \in [0, T]$ and $m \in \mathbb{N}$. Bounding $\|Q(t+\sigma; \bar{Q}, t) - \bar{Q}\|_F$ by $\int_0^\sigma \|\frac{d}{dl}Q(l; \bar{Q}, t)|_{l=t+\sigma}\|_F\, dl$

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and using standard norm properties, we then have

$$
\begin{aligned}
&\int_0^T 1-\operatorname{tr}(M_{12}(t_m+\sigma)\bar{Q})d\sigma \\
&\quad\le \left|\int_0^T 1-\operatorname{tr}(M_{12}(t_m+\sigma)Q(t_m+\sigma;\bar{Q},t_m))\,d\sigma\right| \\
&\qquad+\left|\int_0^T y_{12}^T(t_m+\sigma)(Q(t_m+\sigma;\bar{Q},t_m)-\bar{Q})y_{12}(t_m+\sigma)d\sigma\right| \\
&\quad\le \int_0^T (1-\operatorname{tr}(M_{12}(t_m+\sigma)Q(t_m+\sigma;\bar{Q},t_m)))\,d\sigma+\int_0^T \|Q(t_m+\sigma;\bar{Q},t_m)-\bar{Q}\|_F\,d\sigma \\
&\quad\le T\max_{\sigma\in[0,T]}(1-\operatorname{tr}(M_{12}(t_m+\sigma)Q(t_m+\sigma;\bar{Q},t_m))) \\
&\qquad+\frac{T^2}{2}\max_{\sigma\in[0,T]}\left(\|\frac{d}{dl}Q(l;\bar{Q},t_m)|_{l=t_m+\sigma}\|_F\right) \\
&\quad\le \max_{\sigma\in[0,T]}(TA+T^2\sqrt{A})\quad\text{with}\quad A=1-(y_{12}(t_m+\sigma)^TQ(t_m+\sigma;\bar{Q},t_m)y_{12}(t_m+\sigma)) \\
&\quad\le T\varepsilon+T^2\sqrt{\varepsilon}\,.
\end{aligned}
$$

From the first and last members and B's definition, we need $0=\liminf_{t\to\infty}\frac{1}{T}\int_0^T 1-\operatorname{tr}(M_{12}(t+\sigma)\bar{Q})d\sigma=\liminf_{t\to\infty}\operatorname{tr}(B(t)(I_n-\bar{Q}))$. As symmetric positive semidefinite $B(t)$ has $(n-1)$ eigenvalues lower-bounded by α, this requires $\bar{Q}=I_n$. Therefore (ii) will be satisfied. ∎

Remark 3. (i) It is still unclear whether *exponential* convergence to $\hat{C}_s$ holds in case (b).

(ii) The decrease of $\hat{f}_s\cong f_s$ along solutions is a particularity of $k=2$.

(iii) The freedom on one eigenvalue of $B(t)$ is linked to the particularity that fixing $(n-1)$ column vectors of $Q\in SO(n)$ automatically fixes the last vector; see also the proof of Theorem 2 later. A perhaps more practical condition, although stronger, would be to require $B(t)-\alpha I_n$ positive definite for all $t>0$.

The $\mathbb{R}^n$ dynamics with $k=2$ become $\frac{d}{dt}x=-2x$ (full state coupling) or $\frac{d}{dt}x=-2M_{12}x$ (partial state coupling) with $x=x_2-x_1$. The first system obviously converges to $x=0\Leftrightarrow x_1=x_2$. The partial state coupled system behaves as follows:

- For y_{12} fixed, x globally converges to $(I_n-y_{12}y_{12}^T)\,x$, i.e., the projection of x_2-x_1 onto $\operatorname{span}\{y_{12}\}$ is driven to 0 and the projection of x_1-x_2 onto the orthogonal complement of $\operatorname{span}\{y_{12}\}$ remains constant.
- For varying y_{12}, assuming there exist some fixed $T,\alpha>0$ such that $B(t)-\alpha I_n$ is positive definite for all $t>0$ (thus unlike on $SO(n)$: no rank deficiency allowed here) ensures that x globally converges to 0, implying state synchronization. This standard result actually holds under weaker conditions; see [4]. It is currently not clear how to adapt [4] to $SO(n)$. For the special case where $\frac{d}{dt}y_{12}=\Omega\,y_{12}$ with $\Omega\in\mathfrak{so}(n)$, our setting is equivalent to [39] with particular zero-input dynamics $\frac{d}{dt}x_i=-\Omega\,x_i$.

4. Synchronization conditions for $k>2$ agents. We now turn to the general case of more than two agents. We provide an extensive analysis for fixed y_{ij} in section 4.1 and a result for time-varying y_{ij}, similar to Theorem 1(b), in section 4.2.

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4.1. Fixed reference vectors.

PROPOSITION 1. *The output synchronization set is asymptotically stable under* (6).

Proof. Since (6) is a gradient descent system for f_o and f_o is an analytic function, the set C_o of global minima of f_o is asymptotically stable [1]. ∎

The same holds for (8), the corresponding system in $\mathbb{R}^n$. However, on $SO(n)$, as will be illustrated in section 5, *convergence is not necessarily exponential* for $k > 2$.

4.1.1. Necessary conditions for state synchronization. To reach state synchronization, each agent's state must be observable by combining information on all edges incident to the agent. This necessary condition formalized in Theorem 2 can be extended to the following: considering any partition of $\mathcal{V}$ into $\mathcal{V}_1$ and $\mathcal{V}_2$, the information on edges connecting $\mathcal{V}_1$ to $\mathcal{V}_2$ must make a full state $\in SO(n)$ observable between the two subsets, i.e., the cut must contain at least $n-1$ links with linearly independent reference vectors. Later we show that such necessary condition is *not* sufficient (at all) for $C_s = C_o$.

THEOREM 2. *The* output map *from state space* $(SO(n))^k$ *(resp.,* $(\mathbb{R}^n)^k$*) to output space* $(S^{n-1})^{2\#\mathcal{E}}$ *(resp.,* $\mathbb{R}^{2\#\mathcal{E}}$*) is injective if and only if each* $i \in \mathcal{V}$ *has at least* $n-1$ *(resp.,* n*) linearly independent* y_{ij} *with* $a_{ij} = 1$*. If injectivity is not satisfied, then for any* $(Q_1, \ldots, Q_k) \in C_o$ *there is a continuum of states in* $C_o \setminus C_s$ *giving the same output as* $(Q_1, \ldots, Q_k)$*.*

Proof. We first discuss $SO(n)$. Consider agent i, and renumber the agents such that $a_{ij} = 1$ if and only if $j \le m$. Denote $h_i \colon SO(n) \times (S^{n-1})^m \to (S^{n-1})^m$ the action of Q_i on $\hat{y}_i := (y_{i1}, \ldots, y_{im})$ as $h_i(Q_i, \hat{y}_i) = (Q_i^T y_{i1}, \ldots, Q_i^T y_{im})$. So $Q_i \mapsto h_i(Q_i, \hat{y}_i)$ is the output map of agent i. Define $\mathrm{stab}_h(\hat{y}_i) := \{Q \in SO(n) : h_i(Q, \hat{y}_i) = \hat{y}_i\} = \bigcap_{j=1}^m \mathrm{stab}(y_{ij})$; by definition $y_{i1}, \ldots, y_{im}$ are eigenvectors with eigenvalue 1 of all $Q \in \mathrm{stab}_h(\hat{y}_i) \subseteq SO(n)$. Then $\mathrm{stab}_h(\hat{y}_i) = \{I_n\}$ if and only if $\{y_{i1}, \ldots, y_{im}\}$ contains $(n-1)$ linearly independent vectors. Indeed, the condition implies that all $Q \in \mathrm{stab}_h(\hat{y}_i)$ have $(n-1)$ eigenvalues 1, and $Q \in SO(n)$ implies that the remaining eigenvalue also equals 1; conversely, if there is a ≥ 2-dimensional subspace S_0 orthogonal to $\{y_{i1}, \ldots, y_{im}\}$, then all Q representing rotations in S_0 belong to $\mathrm{stab}_h(\hat{y}_i)$. By invariance of linear independence under a common rotation, similarly a particular output $h_i(Q_i^*, \hat{y}_i)$ of agent i corresponds to a unique state Q_i^* of agent i if and only if $\{y_{i1}, \ldots, y_{im}\}$ contains $(n-1)$ linearly independent vectors. As the whole output map is just the collection of h_i for all $i \in \mathcal{V}$, with h_i depending on the state of agent i only, this proves the first claim. As said above, if the condition does not hold for any particular agent i, then $\mathrm{stab}_h(\hat{y}_i)$ contains rotations in a subspace S_0 that form a Lie subgroup of $SO(n)$ of dimension ≥ 1. If $(Q_1, \ldots, Q_k) \in C_o$, then $(Q_1, \ldots, \hat{Q}Q_i, \ldots, Q_k) \in C_o$ yields the same output for all $\hat{Q} \in \mathrm{stab}_h(\hat{y}_i)$, but at most one $\hat{Q}$ can correspond to state synchronization.

The analog property for $\mathbb{R}^n$ is straightforward: projections on n linearly independent reference vectors specify the state of an agent i completely, while less than n projections leave the possibility to move the state of i in one direction without affecting any of the outputs. ∎

One may be tempted to conclude that injectivity of the output map should guarantee $C_s = C_o$. The following Examples 1 and 2 illustrate that this is not the case. The subtlety is that there exist output values $Q_i^T y_{ij}$ for states in C_o that cannot be obtained with states in C_s. So Theorem 2 only gives a necessary condition. This suggests that a theory of observability in networks is nontrivial.

Example 1. Consider three agents on $SO(3)$ with a fully connected interaction graph and $y_{12} = (1,0,0)^T$, $y_{23} = (0,1,0)^T$, $y_{13} = (0,0,1)^T$. Assumptions of Theo-

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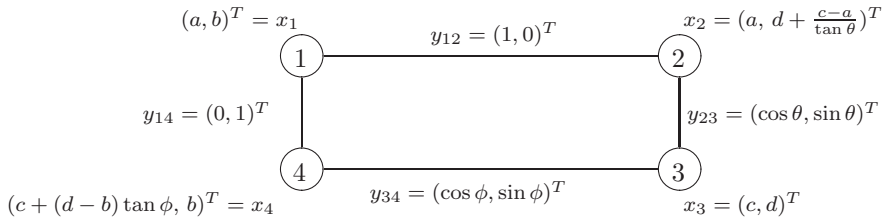

FIG. 1. *Example 2 – each agent sharing its whole state with the network is not sufficient to ensure that output synchronization implies state synchronization.*

rem 2 hold, so each output synchronization value corresponds to a unique point in state space. Let output synchronization be reached at $Q_1^T y_{12} = Q_2^T y_{12} = (1,0,0)^T$, $Q_1^T y_{13} = Q_3^T y_{13} = (0,0,1)^T$, and $Q_2^T y_{23} = Q_3^T y_{23} = (0,-1,0)^T$. The corresponding unique state is $Q_1 = \mathrm{diag}(1,1,1)$, $Q_2 = \mathrm{diag}(1,-1,-1)$, $Q_3 = \mathrm{diag}(-1,-1,1)$, which is not state synchronization.

Example 2. Consider four agents with states $x_i \in \mathbb{R}^2$, where $\mathcal{E} = \{(1,2),(1,4),(2,3),(3,4)\}$, and with $y_{12} = (1,0)^T$, $y_{14} = (0,1)^T$, $y_{23} = (\cos\theta, \sin\theta)^T$, $y_{34} = (\cos\phi, \sin\phi)^T$, assuming $\theta, \phi \notin \{k\pi/2 : k \in \mathbb{Z}\}$ and $y_{23} \neq \pm y_{34}$. The setting is illustrated in Figure 1. Assumptions of Theorem 2 hold, so the output map is injective.[2] Nevertheless, states of the form $x_1 = (a,b)^T$, $x_2 = (a,\ d + \frac{c-a}{\tan\theta})^T$, $x_3 = (c,d)^T$, $x_4 = (c + (d-b)\tan\phi,\ b)^T$ correspond to output synchronization for any $(a,b,c,d) \in \mathbb{R}^4$, while state synchronization holds only if $c = a$ and $d = b$.

PROPOSITION 2. *C_s cannot be asymptotically stable under partial state coupling if $\#\mathcal{E} < \lceil \frac{(k-1)n}{2} \rceil$ for $SO(n)$, if $\#\mathcal{E} < (k-1)n$ for $\mathbb{R}^n$.*

Proof. Equality of outputs on a single link imposes $n-1$ (resp., 1) linear constraints for agents on $kn\frac{n-1}{2}$-dimensional $SO(n)^k$ (resp., on nk-dimensional $(\mathbb{R}^n)^k$). In order for output equalities to constrain the state to $n\frac{n-1}{2}$-dimensional (resp., n-dimensional) C_s, it is thus necessary to impose at least $(k-1)n/2$ (resp., $(k-1)n$) of them. ∎

Proposition 2 is in some sense stronger than Theorem 2, which requires only $k(n-1)/2$ edges for $SO(n)$ (resp., $kn/2$ for $\mathbb{R}^n$). It explains the trouble encountered in Example 2 but does not cover Example 1. However, Proposition 2 does not take the repartition of edges and the values of y_{ij} into account and in this sense is less precise than Theorem 2.

4.1.2. Sufficient conditions for state synchronization. The output cost functions can be written as a generalization of the Laplacian-based quadratic form often used in the context of synchronization algorithms in $\mathbb{R}$. Define $W \in \mathbb{R}^{n(\#\mathcal{E})\times n(\#\mathcal{E})}$ block diagonal with $M_{ij} \in \mathbb{R}^{n\times n}$ in the block corresponding to edge (i,j). Then

$$f_o(Q_1,\ldots,Q_k) = \frac{1}{2}\left(\mathrm{vec}\begin{pmatrix} Q_1 \\ \vdots \\ Q_k \end{pmatrix}\right)^T (I_n \otimes L^g) \left(\mathrm{vec}\begin{pmatrix} Q_1 \\ \vdots \\ Q_k \end{pmatrix}\right), \tag{13}$$

$$g_o(x_1,\ldots,x_k) = \frac{1}{2}\begin{pmatrix} x_1 \\ \vdots \\ x_k \end{pmatrix}^T L^g \begin{pmatrix} x_1 \\ \vdots \\ x_k \end{pmatrix} \tag{14}$$

with generalized Laplacian matrix $L^g = (B \otimes I_n)\, W\, (B \otimes I_n)^T$, where B is the incidence

[2]The generalized necessary condition for $C_s = C_o$ discussed before Theorem 2 is also satisfied.

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matrix of $\mathcal{G}$. In other words, scalar a_{ij} from L is replaced by matrix M_{ij} in L^g. From (13), (14), the rank of L^g expresses how many degrees of freedom the multiagent network can move without affecting f_o or g_o, thus, e.g., without leaving C_o. Standard rank properties yield

$$\operatorname{rank}(L^g) \leq \min(\operatorname{rank} W,\ n \operatorname{rank} B) = \min(\#\mathcal{E},\ n \operatorname{rank} L)\,. \tag{15}$$

The rank of L^g is thus limited by number of edges and graph connectivity: for a graph on k vertices with m connected components, $\operatorname{rank} L = k - m$. However (15) is far from being tight, as it contains no information about the y_{ij}: if, e.g., all y_{ij} are equal, then $\operatorname{rank} L^g = \operatorname{rank} L \leq (k-1)$. For agents in $\mathbb{R}^n$, L^g taking its maximal possible rank is necessary and sufficient for global, exponential state synchronization.

THEOREM 3. *Consider partial state coupling in $\mathbb{R}^n$ with* (8). *$C_o = C_s$ if and only if* $\operatorname{rank} L^g = n(k-1)$. *Moreover, if* $\operatorname{rank} L^g = n(k-1)$, *then the set $C_o = C_s$ is globally exponentially stable.*

Proof. Dynamics (8) write $\frac{d}{dt}x = -L^g x$ with $x = (x_1^T, \ldots, x_k^T)^T \in \mathbb{R}^{nk}$. As L^g is positive semidefinite, x converges to $\ker(L^g) = C_o$; the latter is stable by Proposition 1. Since $C_s \subseteq C_o = \ker(L^g)$ and C_s has dimension n, we have $C_o = C_s$ if and only if $\operatorname{rank} L^g = n(k-1)$. Convergence is exponential and global by linearity. ∎

Note that $\operatorname{rank} L^g = n(k-1)$ requires at least $n(k-1)$ edges and thus $k \geq 2n$ agents; see (15). This minimal number of edges required to reach the sufficient condition of Theorem 3 matches the necessary condition of Proposition 2. Nevertheless, the following example shows that satisfying necessary conditions of both Proposition 2 and Theorem 2 is not sufficient to ensure state synchronization.

Example 3. Consider a network of 20 agents in $\mathbb{R}^2$. $\mathcal{G}$ is set up as follows. The group of agents $\{1, 2, \ldots, 5\}$ is all-to-all connected, idem for the groups $\{6, 7, \ldots, 10\}$, $\{11, 12, \ldots, 15\}$, and $\{16, 17, \ldots, 20\}$; in addition $\mathcal{E}$ contains $(5, 6)$, $(10, 11)$, $(15, 16)$, $(20, 1)$. The y_{ij} are selected all pairwise linearly independent to maximize potential rank of L^g. The generalized Theorem 2 condition holds, as breaking a single link cannot disconnect the system. Proposition 2 holds as $\#E = 44 > n(k-1) = 38$. Nevertheless, the rank of L^g does not exceed $36 < 38$.

A similar treatment can be given on $SO(n)$, but with some major differences: (i) the rank condition is only sufficient; (ii) in particular, nonexponential convergence is possible; (iii) state synchronization is usually not global, even for a complete graph satisfying the rank condition—see section 4.1.3.

THEOREM 4. *Consider partial state coupling on $SO(n)$ with dynamics* (6).

(a) *If* $\operatorname{rank} L^g = n(k-1)$, *then $C_o = C_s$ and the set $C_o = C_s$ is locally exponentially stable.*

(b) *Let column vectors of $V \in \mathbb{R}^{n^2k \times kn(n-1)/2}$ form an orthonormal basis of* $\operatorname{vec}(\mathfrak{so}(n)^k) \subseteq \mathbb{R}^{n^2k}$. *$C_s$ is locally exponentially stable if and only if* $\operatorname{rank}(V^T(I_n \otimes L^g)V) = (k-1)\frac{n(n-1)}{2}$.

Nota Bene: Nonexponential asymptotic stability of C_s is still possible if the rank condition fails.

Proof. Extend $f_o(X_1, \ldots, X_k)$ to $X_i \in \mathbb{R}^{n\times n}$ and let S the subspace of global minima of f_o on $(\mathbb{R}^{n\times n})^k$. S contains C_o, and the subspace $C_s^r := \{(X, \ldots, X) : X \in \mathbb{R}^{n\times n}\}$. If $\operatorname{rank}(L^g) = n(k-1)$, then $\operatorname{rank}(I_n \otimes L^g) = n^2k - n^2$, and from (13), S is n^2-dimensional, like C_s^r, thus $S = C_s^r$. Then $C_o \subset S = C_s^r$ and the intersection with $(SO(n))^k$ yields $C_o = C_s$.

We apply Lemma 1 from the appendix. For $(\hat{Q}, \ldots, \hat{Q}) \in C_s$, we have $L^{gg} = (I_k \otimes \hat{Q}^T)L^g(I_k \otimes \hat{Q})$ with L^{gg} as defined in Lemma 1. Thanks to right-invariance

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w.r.t. absolute orientation, we can reduce the investigation to $\hat{Q} = I_n$ without loss of generality, so $L^{gg} = L^g$. From the previous paragraph, if $\operatorname{rank} L^g = n(k-1)$, then $I_n \otimes L^g$ is positive definite on the subspace of $\mathbb{R}^{kn^2}$ orthogonal to $\operatorname{vec}(C_s^r)$, and thus in particular on the subspace of $\operatorname{vec}(\mathfrak{so}(n)^k)$ orthogonal to $\operatorname{vec}(C_s^r)$. So the condition of Lemma 1 is satisfied and we get exponential stability of C_o, proving (a). By definition, $L_V := V^T(I_n \otimes L^g)V$ is the restriction of $(I_n \otimes L^g)$ to $\operatorname{vec}(\mathfrak{so}(n)^k)$. Obviously, $\operatorname{vec}(C_s^r) \cap \operatorname{vec}(\mathfrak{so}(n)^k) = \{\operatorname{vec}((\Omega^T, \dots, \Omega^T)^T) : \Omega \in \mathfrak{so}(n)\} \subseteq \ker(L_V)$. Since the dimension of $\operatorname{vec}(C_s^r) \cap \operatorname{vec}(\mathfrak{so}(n)^k)$ is $\frac{n(n-1)}{2}$, it follows that L_V is full rank in the subspace of $\operatorname{vec}(\mathfrak{so}(n)^k)$ orthogonal to $\operatorname{vec}(C_s^r)$ if and only if L_V has rank $\frac{kn(n-1)}{2} - \frac{n(n-1)}{2} = (k-1)\frac{n(n-1)}{2}$, proving (b). ∎

Conditions (a) and (b) of Theorem 4 can be viewed as complementary. Checking (a) involves $L^g \in \mathbb{R}^{nk \times nk}$ directly, while (b) involves a larger $n^2k \times n^2k$ matrix on which the roles of graph connectivity and reference vectors are less visible. Condition (b) examines the rank of $(I_n \otimes L^g)$ on the tangent space of $(SO(n))^k$ and therefore is tighter than (a) which basically checks it in $\mathbb{R}^{n^2k \times n^2k}$. Condition (b) can even prove local exponential stability of C_s in situations where $C_o \neq C_s$. If condition (b) fails, C_s can still be asymptotically stable, although convergence cannot be exponential. Section 5 contains examples illustrating these properties for $SO(3)$.

Example 4. On $SO(2) \equiv S^1$, an output $Q_i^T y_{ij}$ is equivalent to full state knowledge: explicitly writing (6) and (4) in terms of rotation angles yields the same dynamics up to gain factor 2. It is in fact equivalent to Kuramoto-like dynamics $\frac{d}{dt}\theta_i = \sum a_{ij} \sin(\theta_j - \theta_i)$ for which local (sometimes almost-global) state synchronization holds as soon as $\mathcal{G}$ is connected; see, e.g., [34]. There are many synchronizing situations where $\operatorname{rank}(L^g) < 2(k-1)$, e.g., when $\mathcal{G}$ has less than $2(k-1)$ edges, illustrating that Theorem 4(a) gives no necessary condition. In contrast, Theorem 4(b) perfectly captures the situation on $SO(2)$: calculations with V constructed, e.g., from basis elements $(\dots, \left(\begin{smallmatrix} 0 & 0 \\ 0 & 0 \end{smallmatrix}\right), \left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right), \left(\begin{smallmatrix} 0 & 0 \\ 0 & 0 \end{smallmatrix}\right), \dots)$ of $\mathfrak{so}(2)^k$—the nonzero matrix running over all k positions in the list—show that actually $V^T(I_n \otimes L^g)V = L$.

4.1.3. About almost-global convergence. On $\mathbb{R}^n$, stability of state synchronization is always global; see Theorem 3. On $SO(n)$ with full state exchange (see [34]), C_s is not always (almost-)globally asymptotically stable, but (4) has no other stable equilibria than C_s when $\mathcal{G}$ is, e.g., all-to-all or a tree. For partial state coupling on $SO(n)$, $n > 2$, (6) does not retain these global properties[3] for trees and complete graphs. For a tree, the number of links is insufficient to share all state information: the necessary condition from Theorem 2 is not satisfied. The following example illustrates that even with all-to-all interaction and conditions of Theorem 4 holding, stable equilibria besides the state consensus set can exist.

Example 5. Consider $k = 7$ agents on $SO(3)$, interconnected with $y_{ij} = (0, 1, 0)^T$ if $(i-j) \bmod 7 = \pm 1$, $y_{ij} = (0, 0, 1)^T$ if $(i-j) \bmod 7 = \pm 3$, and $y_{ij} = (\sin(\alpha), 0, \cos(\alpha))^T$ if $(i-j) \bmod 7 = \pm 2$ with $\alpha = 0.04$. This implies $\operatorname{rank}(L^g) = 18 = n(k-1)$, so Theorem 4 applies and $C_s = C_o$ is locally exponentially stable. Nevertheless, (6) admits an equilibrium $\notin C_s$ of the form $Q_i = Q_y Q_z(i\frac{2\pi}{7})$, $i = 1, \dots, 7$, with

$$Q_z(\theta) = \begin{pmatrix} \cos(\theta) & \sin(\theta) & 0 \\ -\sin(\theta) & \cos(\theta) & 0 \\ 0 & 0 & 1 \end{pmatrix}, \qquad Q_y = \begin{pmatrix} \cos(\beta) & 0 & \sin(\beta) \\ 0 & 1 & 0 \\ -\sin(\beta) & 0 & \cos(\beta) \end{pmatrix},$$

[3] Contrary to an erroneous statement in [19, Proposition 2(b)].

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and $\beta = -0.0558\ldots$. Checking the Hessian at that point shows that it is stable. The small but nonzero values of α and consequently β are necessary to have at the same time L^g of maximal rank and stability of the equilibrium.

4.2. Time-varying reference vectors. Time-varying y_{ij} are favorable for state synchronization, as they give access to different partial state information over time. We assume that the y_{ij} are smooth functions of time $\mathbb{R} \to S^{n-1}$. Unlike for $k = 2$ agents, f_s does not decrease along solutions of (6) in general. This somewhat complicates the analysis and motivates us to introduce a scaling parameter ε in (6), i.e.,

$$\frac{d}{dt} Q_i = \varepsilon\, Q_i \sum_{\{j:(i,j)\in\mathcal{E}\}} \operatorname{skew}(Q_i^T M_{ij}(t)\, Q_j), \quad i = 1, 2, \ldots, k\,. \tag{16}$$

To show convergence to C_s, we need a persistent excitation condition on the outputs.

Assumption 1. $\overline{M}_{ij} := \lim_{t\to\infty} \frac{1}{2t}\int_{-t}^{t} M_{ij}(s)\, ds$ exists and has at most one zero eigenvalue for each $(i,j) \in \mathcal{E}$. Moreover,[4] there exists $Z : [0, +\infty) \to [0, +\infty)$ with $\lim_{t\to\infty} \frac{1}{t} Z(t) = 0$ and such that for all t and $(i,j) \in \mathcal{E}$, it holds that $\| \int_t^{t+T} (M_{ij}(s) - \overline{M}_{ij})\, ds\|_F \le Z(T)$ for all $T \ge 0$.

Assumption 1 allows to consider the "average system" (for $\varepsilon = 1$)

$$\frac{d}{dt} \overline{Q}_i = \overline{Q}_i \sum_{\{j:(i,j)\in\mathcal{E}\}} \operatorname{skew}(\overline{Q}_i^T \overline{M}_{ij} \overline{Q}_j) \tag{17}$$

which converges like the full state coupling case (4); see [34] for full state coupling convergence properties. For sufficiently small ε, an averaging argument shows that this convergence extends to time-varying dynamics (16).

PROPOSITION 3. *If $\mathcal{G}$ is connected and Assumption* 1 *holds, then C_s is locally exponentially stable for* (17).

Proof. Equation (17) is a gradient system for the average cost $\overline{f}(Q_1, \ldots, Q_k) = \frac{1}{2}\sum_{(i,j)\in\mathcal{E}}^{k} \|\overline{M}_{ij}(Q_i - Q_j)\|_F^2$. A straightforward extension of the arguments in [34] yields that C_s is locally asymptotically stable, because C_s is an isolated minimum set of cost function $\overline{f}$. Exponential stability holds similarly to other results in the paper. ∎

THEOREM 5. *If $\mathcal{G}$ is connected and Assumption* 1 *holds, then for sufficiently small $\varepsilon > 0$, C_s is locally exponentially stable for* (16).

Proof. We first project the system onto one where C_s is represented by a single point. For this we use reductive homogeneous spaces; see, e.g., [9].

C_s is a closed subgroup of $SO(n)^k$. Define the compact homogeneous space $\mathcal{M} = SO(n)^k/C_s$, with C_s acting on $SO(n)^k$ by right multiplication, and canonical projection $\pi\colon SO(n)^k \to \mathcal{M}$. The image of C_s under π is a point $p \in \mathcal{M}$. Since $SO(n)^k$ is compact, $\mathcal{M}$ is a reductive homogeneous space. Equip $\mathcal{M}$ with the normal Riemannian metric induced by the product metric on $SO(n)^k$. Denote $F_{SO}(Q_1, \ldots, Q_k, t)$ and $\overline{F}_{SO}(Q_1, \ldots, Q_k)$, respectively, the vector fields corresponding to (16) with $\varepsilon = 1$ and to (17). Thanks to invariance of the dynamics under right multiplication of $SO(n)^k$ by C_s, they respectively induce vector fields $F(x,t)$ and $\overline{F}(x)$ on $\mathcal{M}$, where $\overline{F}$ is the time average of F. Moreover, for both time-varying and time-invariant dynamics, point $p \in \mathcal{M}$ has the same stability properties as set $C_s \subset SO(n)^k$. So p is exponentially stable under $\overline{F}$ and we must prove that this remains true under $\varepsilon F(x,t)$ for ε small. We therefore apply Proposition 3 in [3].

[4] This condition of uniform convergence to $\overline{M}_{ij}$ was not acknowledged in [19].

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Since all necessary regularity conditions for Proposition 3 of [3] hold, it remains to examine $\| \int_{t_1}^{t_2} F(x,t) - \overline{F}(x)\, dt\|$ for any fixed $x \in \mathcal{M}$ in a neighborhood of p,

$$\left\| \int_{t_1}^{t_2} F(x,t) - \overline{F}(x)\, dt \right\| \leq \left\| \int_{t_1}^{t_2} F_{SO}(Q_1,\dots,Q_k,t) - \overline{F}_{SO}(Q_1,\dots,Q_k)\, dt \right\|,$$

by projection from $T_{Q_1,\dots,Q_k} SO(n)^k$ to $T_x\mathcal{M}$ with $\pi(Q_1,\dots,Q_k) = x$. Now

$$\begin{aligned} &\left\| \int_{t_1}^{t_2} F_{SO}(Q_1,\dots,Q_k,t) - \overline{F}_{SO}(Q_1,\dots,Q_k)\, dt \right\|^2 \\ &\quad = \sum_{i=1}^{k} \left\| \sum_{j=1}^{k} a_{ij} \operatorname{skew}\left(Q_i^T \int_{t_1}^{t_2} (M_{ij}(t) - \overline{M}_{ij})\, dt\, Q_j \right) \right\|_F^2 \\ &\quad \leq k^2 \sum_{i=1}^{k}\sum_{j=1}^{k} a_{ij} \left\| \operatorname{skew}\left(Q_i^T \int_{t_1}^{t_2} (M_{ij}(t) - \overline{M}_{ij})\, dt\, Q_j \right) \right\|_F^2 . \end{aligned} \tag{18}$$

Using $\| \operatorname{skew}(Q_i^T P Q_j)\|_F = \frac{1}{2}\|P(I_n - Q_jQ_i^T) - (I_n - Q_jQ_i^T)^T P\|_F \leq \|P\|_F\, \|I_n - Q_jQ_i^T\|_F = \|P\|_F\, \|Q_i - Q_j\|_F$ for any symmetric $P \in \mathbb{R}^{n\times n}$, and Assumption 1, we get

$$\begin{aligned} [\text{Eq.}(18)] &\leq k^2 \sum_{i=1}^{k}\sum_{j=1}^{k} a_{ij} \left\| \int_{t_1}^{t_2} (M_{ij}(t) - \overline{M}_{ij})\, dt \right\|^2 \|Q_i - Q_j\|_F^2 \\ &\leq k^2\, Z(t_2 - t_1)^2 \sum_{i=1}^{k}\sum_{j=1}^{k} a_{ij}\, \|Q_i - Q_j\|_F^2 \\ &\leq\ k^2 Z(t_2 - t_1)^2 \sum_{i=1}^{k}\sum_{j=1}^{k} \|Q_i - Q_j\|_F^2. \end{aligned}$$

Since $\sum_{i,j=1}^{k} \|Q_i - Q_j\|_F^2$ is a quadratic form on $(\mathbb{R}^{n\times n})^k$ with kernel $C_s^r = \{(X,\dots,X) : X \in \mathbb{R}^{n\times n}\}$, there is a constant $c > 0$ such that with $\operatorname{dist}_{(\mathbb{R}^{n\times n})^k}$ the Euclidean distance, $\sum_{i,j=1}^{k} \|Q_i - Q_j\|_F^2 \leq c \operatorname{dist}^2_{(\mathbb{R}^{n\times n})^k}((Q_1,\dots,Q_k), C_s^r)$. As $C_s \subset C_s^r$ and the Riemannian metric on $SO(n)$ is induced by the Euclidean metric on $\mathbb{R}^{n\times n}$, we have $\operatorname{dist}_{(\mathbb{R}^{n\times n})^k}((Q_1,\dots,Q_k), C_s^r) \leq \operatorname{dist}_{SO(n)^k}((Q_1,\dots,Q_k), C_s)$ with $\operatorname{dist}_{SO(n)^k}$ the Riemannian distance on $SO(n)^k$. This yields $\sum_{i,j=1}^{k} \|Q_i - Q_j\|_F^2 \leq c \operatorname{dist}^2_{SO(n)^k}((Q_1,\dots,Q_k), C_s) = c \operatorname{dist}^2_{\mathcal{M}}(x,p)$ with $p = \pi(C_s)$. Then overall, $\| \int_{t_1}^{t_2} F(x,t) - \overline{F}(x)\, dt\| \leq k\sqrt{c}\, Z(t_2 - t_1) \operatorname{dist}_{\mathcal{M}}(x,p)$. Using local charts around p, this shows that the last condition of Proposition 3 in [3] holds, ensuring local exponential stability of p under $F(x, \frac{1}{\varepsilon}t)$ for ε sufficiently small. We conclude by change of timescale: $\frac{d}{dt}x = \varepsilon\, F(x,t)$ is equivalent to $\frac{d}{d\tau}x = F(x, \frac{1}{\varepsilon}\tau)$. $\square$

The condition on ε can be understood as follows: the smaller ε, the more information (different $y_{ij}(t)$) the system gathers w.r.t. its motion in a fixed time span, and thus the closer it can follow a full state knowledge evolution (according to $\overline{M}_{ij}$). It is possible to give quantitative estimates for ε based on results in [3], but these technical aspects are not the focus of the present paper.

Remark 4. Adding edges is never detrimental for local consensus in the average system linearized around C_s: if C_s is an isolated local minimum of the associated cost function, then adding edges does not modify this property. Therefore if local

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exponential convergence is proved for a subgraph of $\mathcal{G}$, it also holds for $\mathcal{G}$. As a consequence, Theorem 5 in fact requires the rank condition on $\overline{M}_{ij}$ in Assumption 1 to hold only on the edges of a spanning tree of $\mathcal{G}$. It should also be possible to extend the result to networks where a majority of fixed y_{ij} are combined with a few time-varying y_{ij} that enhance the connection on particular links: it suffices to show that C_s is locally exponentially stable for the average system, and then the conclusion of Theorem 5 holds.

5. State synchronization on $SO(3)$. We pursue the analysis for fixed y_{ij} on special case $SO(3)$, which has practical importance as describing rotations in three-dimensional space. It is also the simplest nontrivial output consensus setting. $Q_i^T y_{ij} = Q_j^T y_{ij}$ fixes $Q_i Q_j^T$ up to a rotation of axis y_{ij}; the equation indeed contains two-dimensional information about three-dimensional relative errors $Q_i Q_j^T \in SO(3)$. We give further conditions to have $C_o = C_s$, study error sensitivity highlighting poor robustness when y_{ij} are relative positions in $\mathbb{R}^n$, and illustrate with simulations.

5.1. Weaker conditions for $C_o = C_s$. C_o and C_s are static properties (involving the state and output map only), while stability of local equilibria depends on the specific coupling law. Changing the coupling law—by, e.g., changing the norm chosen in f_o, the order of the dynamics, or introducing stochasticity—can change stability of points. One might thereby find ways (see [33]) to avoid nonglobal minima of f_o and get only C_o stable. Such investigation is especially worthwhile if $C_o = C_s$, since one might hope to make C_s the only stable set. In contrast, if $C_o \neq C_s$, states in $C_o \setminus C_s$ are unavoidably locally stable, whatever coupling alternative is used to almost-globally stabilize C_o. This justifies our further interest in characterizing situations where $C_s = C_o$. Another reason to investigate when $C_s = C_o$ is that, although the local convergence to C_o highlighted in section 4.1.3 appears to have a rather large basin of attraction, C_s would not inherit that property if there are "spurious" $C_o \setminus C_s$ states close to C_s.

The aim of this section is to give network characteristics, possibly useful for design, that ensure $C_s = C_o$ on $SO(3)$ with generic reference vectors y_{ij}. We first gain some insight on small networks (Theorem 6), which is then used in an algorithm that handles arbitrarily large networks (Corollary 1).

THEOREM 6. *Consider k agents applying* (6) *on $SO(3)$ with fixed y_{ij}.*

(a) *If $k = 3$, $\mathcal{E} = \{(1,2),(1,3),(2,3)\}$, and* $\operatorname{rank}(y_{12}, y_{13}, y_{23}) = \operatorname{rank}(y_{12}, y_{13}) = \operatorname{rank}(y_{12}, y_{23}) = \operatorname{rank}(y_{13}, y_{23}) = 2$, *then $C_o = C_s$.*

(b) *If $k = 3$, $\mathcal{E} = \{(1,2),(1,3),(2,3)\}$, and* $\operatorname{rank}(y_{12}, y_{13}, y_{23}) = 3$, *then $C_o \neq C_s$. However, output synchronization still implies state synchronization* locally *near C_s.*

(c) *If $k = 4$, $\mathcal{E} \supseteq \{(1,2), (1,3), (1,4), (2,3), (2,4)\}$,* $\operatorname{rank}(y_a, y_b, y_c) = 3$ *for each triple of different $a, b, c \in \mathcal{E}$, and* $\frac{(y_{12}\times y_{23})^T(y_{12}\times y_{13})}{y_{23}^T(y_{12}\times y_{13})} \neq \frac{(y_{12}\times y_{24})^T(y_{12}\times y_{14})}{y_{24}^T(y_{12}\times y_{14})}$, *then $C_o = C_s$.*

Proof. Define relative states $R_j = Q_j Q_1^T$, $j = 2, \ldots, k$. State synchronization means $R_j = I_3$, $j = 2, \ldots, k$. For (a) and (b), output synchronization means $R_2 y_{12} = y_{12}$, $R_3 y_{13} = y_{13}$, and $R_3 R_2^T y_{23} = y_{23}$. For (c), output synchronization requires these on subnetwork A of agents $\{1,2,3\}$ and the same conditions on subnetwork B of agents $\{1,2,4\}$.

(a) Output synchronization implies $y_{12}^T R_3^T y_{23} = y_{12}^T R_3^T R_3 R_2^T y_{23} = y_{12}^T y_{23}$. The rank conditions yield $y_{23} = a y_{12} + b y_{13}$ with $a, b \in \mathbb{R} \setminus \{0\}$. Thus $y_{12}^T R_3^T y_{23} - y_{12}^T y_{23} = a(y_{12}^T R_3 y_{12} - 1) = 0$, which implies $R_3 y_{12} = y_{12}$. Then R_3 has two linearly independent eigenvectors for the eigenvalue 1, hence $R_3 = I_3$ (see proof of Theorem 2). A similar argument yields $R_2 = I_3$.

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(b) For any $x \in S^2$ we have $\{Rx : R \in \mathrm{stab}(y_{12})\} = \{y \in S^2 : y_{12}^T y = y_{12}^T x\}$, and for $x \neq \pm y_{12}$ there is a bijection between R and y. The rank conditions allow us to write $y_{23} = a_1\mathbf{e}_1 + a_2\mathbf{e}_2 + a_3\mathbf{e}_3$ in the orthonormal frame $(\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3) := (y_{12}, \frac{y_{12}\times y_{13}}{\|y_{12}\times y_{13}\|_2}, \frac{y_{12}\times(y_{12}\times y_{13})}{\|y_{12}\times y_{13}\|_2})$ with $a_2 \neq 0$. Define $y_* = a_1\mathbf{e}_1 - a_2\mathbf{e}_2 + a_3\mathbf{e}_3$. Since $y_{12}^T y_* = y_{12}^T y_{23} \neq \pm 1$, there is a unique $R_{2*} \in \mathrm{stab}(y_{12}) \setminus I_3$ such that $R_{2*}^T y_{23} = y_*$. Similarly, using $y_{13}^T(y_{12} \times y_{13}) = 0$, there is a unique $R_{3*} \in \mathrm{stab}(y_{13}) \setminus I_3$ such that $R_{3*}y_{23} = y_*$. Then $R_{3*}R_{2*}^T y_{23} = y_{23}$, so R_{2*} and R_{3*} differ from I_3 but satisfy all conditions of output synchronization, i.e., $C_o \setminus C_s \neq \emptyset$. Furthermore, states in $C_o \setminus C_s$ all correspond to the same relative state (R_{2*}, R_{3*}). Indeed, assume $y_\circ = R_{2\circ}^T y_{23} = R_{3\circ}^T y_{23}$ for some $(R_{2\circ}, R_{3\circ}) \in \mathrm{stab}(y_{12}) \times \mathrm{stab}(y_{13})$, characterizing output synchronization. The linearly independent requirements $y_{12}^T y_\circ = y_{12}^T y_{23}$ and $y_{13}^T y_\circ = y_{13}^T y_{23}$ have two solutions in S^2: y_{23} and y_*. As each admissible $y_\circ$ maps to unique $(R_{2\circ}, R_{3\circ}) \in \mathrm{stab}(y_{12}) \times \mathrm{stab}(y_{13})$, the only possible $(R_{2\circ}, R_{3\circ})$ are (I_3, I_3) and (R_{2*}, R_{3*}). Thus C_o consists of two connected components: C_s and $C_o \setminus C_s = \{(Q, R_{2*}Q, R_{3*}Q) : Q \in SO(3)\}$.

(c) From case (b), output synchronization in subnetwork $A = \{1, 2, 3\}$ requires $(R_2, R_3) \in \{(R_{2*}, R_{3*}), (I_3, I_3)\}$ with $I_3 \notin \{R_{2*}, R_{3*}\}$. Similarly we need $(R_2, R_4) \in \{(R_{2\bigstar}, R_{4*}), (I_3, I_3)\}$ with $I_3 \notin \{R_{2\bigstar}, R_{4*}\}$ for output synchronization in subnetwork $B = \{1, 2, 4\}$; hereafter we characterize $(R_{2\bigstar}, R_{4*})$ by $y_\bigstar \in S^2$. Output synchronization in the whole network combines these conditions. Then consistency for R_2 requires either $R_2 = I_3$ (implying $R_3 = R_4 = I_3$, state synchronization) or $R_2 = R_{2\bigstar} = R_{2*}$. We show that the latter is impossible. Let (a_1, a_2, a_3) coordinates of y_{23} as in (b), in particular $a_2 \neq 0$. Write $y_{24} = b_1\mathbf{e}_1 + b_2\mathbf{e}_4 + b_3\mathbf{e}_5$ in the orthonormal frame $(\mathbf{e}_1, \mathbf{e}_4, \mathbf{e}_5) = (y_{12}, \frac{y_{12}\times y_{14}}{\|y_{12}\times y_{14}\|_2}, \frac{y_{12}\times(y_{12}\times y_{14})}{\|y_{12}\times y_{14}\|_2})$ with $b_2 \neq 0$ by the same argument as in (b). Define $R \in \mathrm{stab}(y_{12})$ with $(\mathbf{e}_1, \mathbf{e}_4, \mathbf{e}_5) = R\,(\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3)$. Note that $R_{2*}^T(y_{23} - a_1\mathbf{e}_1) = (y_* - a_1\mathbf{e}_1) = (a_2\mathbf{e}_2 + a_3\mathbf{e}_3)$ and $R^T R_{2\bigstar}^T(y_{24} - b_1\mathbf{e}_1) = R^T(y_\bigstar - b_1\mathbf{e}_1) = (b_2\mathbf{e}_2 + b_3\mathbf{e}_3)$. Assume $R_{2\bigstar} = R_{2*}$. Then

$$
\begin{aligned}
&(R_{2*}^T(y_{23} - a_1\mathbf{e}_1)) \times (R^T R_{2*}^T(y_{24} - b_1\mathbf{e}_1)) \\
&\quad = (a_2\mathbf{e}_2 + a_3\mathbf{e}_3) \times (b_2\mathbf{e}_2 + b_3\mathbf{e}_3) = m\mathbf{e}_1 = R_{2*}^T\, m\mathbf{e}_1 \\
&\quad = R_{2*}^T\,((a_2\mathbf{e}_2 + a_3\mathbf{e}_3) \times (b_2\mathbf{e}_2 + b_3\mathbf{e}_3)) \\
&\quad = (R_{2*}^T(a_2\mathbf{e}_2 + a_3\mathbf{e}_3)) \times (R_{2*}^T(b_2\mathbf{e}_2 + b_3\mathbf{e}_3)) = (-a_2\mathbf{e}_2 + a_3\mathbf{e}_3) \times (-b_2\mathbf{e}_2 + b_3\mathbf{e}_3).
\end{aligned}
$$

The first and last expressions yield $\frac{a_3}{a_2} = \frac{b_3}{b_2}$. Expressing a_2, a_3, b_2, b_3 by scalar products leads to $\frac{(y_{12}\times y_{23})^T(y_{12}\times y_{13})}{y_{23}^T(y_{12}\times y_{13})} = \frac{(y_{12}\times y_{24})^T(y_{12}\times y_{14})}{y_{24}^T(y_{12}\times y_{14})}$, violating the last assumption of the statement. Thus $R_{2\bigstar} = R_{2*}$ is impossible and $C_o \setminus C_s = \emptyset$. $\square$

Remark 5. (i) Rank conditions in Theorem 6(b) and (c) are only violated on a set of measure zero for freely independently chosen $y_{ij} \in S^2$. Situation (a)—coplanar but nonaligned y_{12}, y_{13}, y_{23}—is only violated on a set of measure zero for $y_{ij} = \frac{y_i - y_j}{\pm\|y_i - y_j\|_2}$, e.g., reference vectors derived from relative positions of agents located at freely chosen $y_i \in R^3$. In section 5.2 we show that state synchronization with (a) is ill-conditioned. Simulations tend to indicate that this is characteristic of reference vectors defined as "relative positions."

(ii) Any network on $k = 4$ agents that satisfies the necessary condition of Proposition 2 satisfies the sufficient condition of Theorem 6(c) for freely independently chosen $y_{ij} \in S^2$. Thus conversely, four agents having less interconnection than for Theorem 6(c) cannot reach state synchronization.

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Example 6. The general results of section 4.1 relate to Theorem 6 as follows:

- For the setting of Theorem 6(c), $\operatorname{rank} L^g \leq \#\mathcal{E} = 6 < n(k-1) = 9$, so the condition of Theorem 4(a) is not satisfied. One checks that the condition of Theorem 4(b) holds.
- For the setting of Theorem 6(b), the condition of Theorem 4(a) is a fortiori still not satisfied, but the condition of Theorem 4(b) holds and ensures that C_s is locally asymptotically stable, despite $C_o \neq C_s$.
- Take as an example setting of Theorem 6(a) the situation of Example 1 but with reference vectors $y_{12} = y_{13} = (1,0,0)^T$ and $y_{23} = (0,1,0)^T$. From Theorem 6(a) and Proposition 1, $C_s = C_o$ is locally asymptotically stable. However, the condition of Theorem 4(b) is *not* satisfied, so convergence cannot be exponential.

The condition of Theorem 4(a) appears to be too strong for small networks.

Theorem 6(c) gives a generic condition for $C_o = C_s$ on small graphs. The following algorithm exploits this condition by repeatedly collapsing subnetworks, which have the graph structure as in 6(c), to single nodes. If the resulting graph consists of a single point, then generic reference vectors yield state synchronization.

- `WHILE` $\exists$ `different` $i,l,m,p \in \mathcal{V}$ `with` $(i,l),(i,m),(p,l),(p,m),(m,l) \in \mathcal{E}$
  1. `Collapse` i,l,m,p `to a single vertex` i`---the resulting graph can have several edges connecting the same two vertices;`
  2. `WHILE` $\exists$ `vertices` i,j `connected by more than one edge:`
     `Collapse` i,j `into a single vertex` i`;`
     `ENDWHILE`

  `ENDWHILE`

- `Output the resulting graph;`

COROLLARY 1. *If the above algorithm reduces $\mathcal{G}(\mathcal{V},\mathcal{E})$ to a single vertex, then $C_o = C_s$ for a generic set of freely chosen reference vectors $y_{ij} \in S^2$.*

Proof. The algorithm looks for small subnetworks of $\mathcal{G}$ for which output synchronization implies state synchronization. If agents i,s belong to such a subnetwork, then all their links with the rest of the network are just further constraints that apply to their common state $Q_i = Q_s \in SO(3)$. We can therefore collapse i,s into a single vertex with $Q_i = Q_s$, and connect to this single vertex all the edges that were previously connected to i or s. This operation might lead to a pair of vertices connected by more than one edge. The algorithm therefore looks for subnetworks of two types: (a) four-agent groups matching Theorem 6(c) for generic reference vectors and (b) a pair of agents linked by several edges. In the latter case, generic reference vectors imply (at least) a double constraint $R\,y_{ij}^{(1)} = y_{ij}^{(1)}$ and $R\,y_{ij}^{(2)} = y_{ij}^{(2)}$ on $R = Q_j Q_i^T$, which can only be satisfied with $R = I_3$; thus C_o indeed implies $Q_j = Q_i$ in subnetwork i,j. If the collapsing procedure can be applied until one agent remains, then $C_o = C_s$. ∎

Corollary 1 gives a sufficient condition for $C_o = C_s$. It analyzes $\mathcal{G}$ through small subgraphs and will not cover networks where good connection specifically relies on larger structures. Types of graphs for which Corollary 1 ensures $C_o = C_s$ include (of course adding more edges always keeps $C_o = C_s$) a square grid of at least 3×3 agents where one diagonal edge is added to two adjacent squares, or a three-dimensional cubic grid, adding one face-diagonal edge, or any graph obtained by iteratively adding one node and two edges, starting from the situation of Theorem 6(c). The latter illustrates that the algorithm associated to Corollary 1 may also be used in reverse, to design a $C_o = C_s$-ensuring graph by successive expansions.

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FIG. 2. *Nonrigid graphs for which* (a) $C_o = C_s$ *from Corollary* 1 *and* (b) $C_o \neq C_s$.

Remark 6. The condition expressed by our graph analysis algorithm differs from planar formation rigidity; see, e.g., [12] for a recent study. With three agents, rigidity is a weaker condition as it holds for graphs satisfying the condition of Theorem 6(b). With four agents, a graph satisfying Theorem 6(c) is minimal both for (local) rigidity and for $C_o = C_s$. With more agents, our objective becomes easier to achieve: e.g., the graph in Figure 2(a) ensures $C_o = C_s$ but is clearly not rigid. Note that also *for agent states in* $\mathbb{R}^2$, i.e., dynamics (8), a rigid graph does not ensure $C_o = C_s$ with generic y_{ij}; consider, e.g., three all-to-all connected agents.[5]

5.2. Error sensitivity. Achieving $C_s = C_o$ with fewer agents for coplanar y_{ij} (Theorem 6(a)) than for linearly independent y_{ij} (Theorem 6(b)) may seem surprising. In fact, the two solutions $(Q_2Q_1^T, Q_3Q_1^T) \in \{(I_3, I_3), (R_{2*}, R_{3*})\}$ for output synchronization in the proof of Theorem 6(b) correspond to the intersection of circles $\{R_2 y_{23} : R_2 \in \text{stab}(y_{12})\}$ and $\{R_3 y_{23} : R_3 \in \text{stab}(y_{13})\}$ on S^2. In the coplanar case, the circles are tangent: this leaves (I_3, I_3) as the only solution but also indicates a possible loss of hyperbolicity of the dynamical system, which can have undesirable consequences. The following formalizes this on the basis of error sensitivity. In section 5.3, simulations illustrate the effect on the convergence speed.

We consider measurement errors replacing $Q_i^T y_{ij}$ by

$$Q_i^T E_{ij} y_{ij} \tag{19}$$

with fixed perturbations $E_{ij} \in \{R \in SO(3) : \|R - I_3\|_F < \varepsilon \ll 1\}$. The actual closed-loop evolution with these perturbed measurements,

$$\tfrac{d}{dt} Q_i = Q_i \sum_{\{j:(i,j)\in\mathcal{E}\}} \text{skew}(Q_i^T\, E_{ij} M_{ij} E_{ji}^T\, Q_j), \quad i = 1, 2, \ldots, k\,, \tag{20}$$

is the gradient of $f_{o-e} = \sum_{(i,j)\in\mathcal{E}} (1 - \text{tr}(Q_i^T E_{ij} M_{ij} E_{ji}^T Q_j))$. A priori, $E_{ij} \neq E_{ji}$ breaks the output map symmetry and $Q_i = Q_j$ does not yield synchronization of the perturbed outputs. For clarity, we denote $C_{o-e}(E_{12}, \ldots, E_{k\,(k-1)})$ the output consensus set with perturbed outputs (19). For a large number of edges, f_{o-e} is strictly positive over $(SO(n))^k$ for generic y_{ij} and E_{ij}, meaning that $C_{o-e} = \emptyset$. Given two joint state sets C_A, C_B we define

$$\text{dist}^2(C_A, C_B) := \min_{\substack{(Q_{1A}, Q_{2A}, \ldots, Q_{kA}) \in C_A, \\ (Q_{1B}, Q_{2B}, \ldots, Q_{kB}) \in C_B}} \sum_{i=1}^{k} \|Q_{iA} - Q_{iB}\|_F^2\,.$$

THEOREM 7. *Consider* $k = 3$ *agents applying* (20) *on* $SO(3)$. *Denote by* E_ε *the set* $\{(E_{12}, E_{13}, E_{21}, E_{23}, E_{31}, E_{32}) : \|I_3 - E_{ij}\|_F < \varepsilon$ *for all* $i, j\} \subset (SO(3))^{k(k-1)}$.

[5]This might be a more telling distinction, since planar rigidity also involves agent states in $\mathbb{R}^2$ and scalar equality constraints along edges.

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(a) *In the setting of Theorem* 6(a) *there is a constant* $c > 0$ *such that for any sufficiently small* $\varepsilon > 0$, *there is a nonzero measure set* $S_\varepsilon \subset E_\varepsilon$ *where* $C_{o-e}(E_{12}, \ldots, E_{32}) \neq \emptyset$ *and*

$$\operatorname{dist}(C_s, C_{o-e}(E_{12}, \ldots, E_{32})) \geq c\sqrt{\varepsilon} \quad \textit{for all } (E_{12}, \ldots, E_{32}) \in S_\varepsilon.$$

Furthermore, for any $\varepsilon > 0$ *there is also a nonzero measure set* $N_\varepsilon \subset E_\varepsilon$ *such that* $C_{o-e}(E_{12}, \ldots, E_{32}) = \emptyset$ *for all* $(E_{12}, \ldots, E_{32}) \in N_\varepsilon$.

(b) *In the setting of Theorem* 6(b), *there is a constant* $c > 0$ *such that for any sufficiently small* $\varepsilon > 0$, $C_{o-e}(E_{12}, \ldots, E_{32})$ *is nonempty and*

$$\operatorname{dist}(C_s, C_{o-e}(E_{12}, \ldots, E_{32})) \leq c\,\varepsilon \quad \textit{for all } (E_{12}, \ldots, E_{32}) \in E_\varepsilon.$$

Proof. The simple but tedious proof, constructing specific sets, is in the appendix. ∎

So in the setting of Theorem 6(a), the ratio between state synchronization error $\operatorname{dist}(C_s, C_{o-e})$ and measurement error ε can grow to infinity for $\varepsilon \to 0$, while in the setting of Theorem 6(b) this ratio remains finite; in this sense only the second situation is robust. From geometric considerations (i.e., set intersections; see the proof of Theorem 7), we also expect good robustness in the setting of Theorem 6(c). Simulations suggest that this distinction—y_{ij} freely chosen in S^2 robust, $y_{ij} = (y_i - y_j) / \|y_i - y_j\|_2$ not robust—carries over to Corollary 1.

5.3. Numerical simulations. Figure 3 shows local evolution of f_o and f_s for $k = 6$ all-to-all-connected agents implementing (6). Thick lines are for generic y_{ij} independently freely chosen in S^2 (henceforth denoted B, analogous to case (b) of Theorems 6 and 7); thin lines are for y_{ij} derived from generic positions $y_i \in \mathbb{R}^3$, that is, $y_{ij} = \frac{y_i - y_j}{\pm\|y_i - y_j\|_2}$ (denoted A, analogous to case (a) of the theorems). The top plots

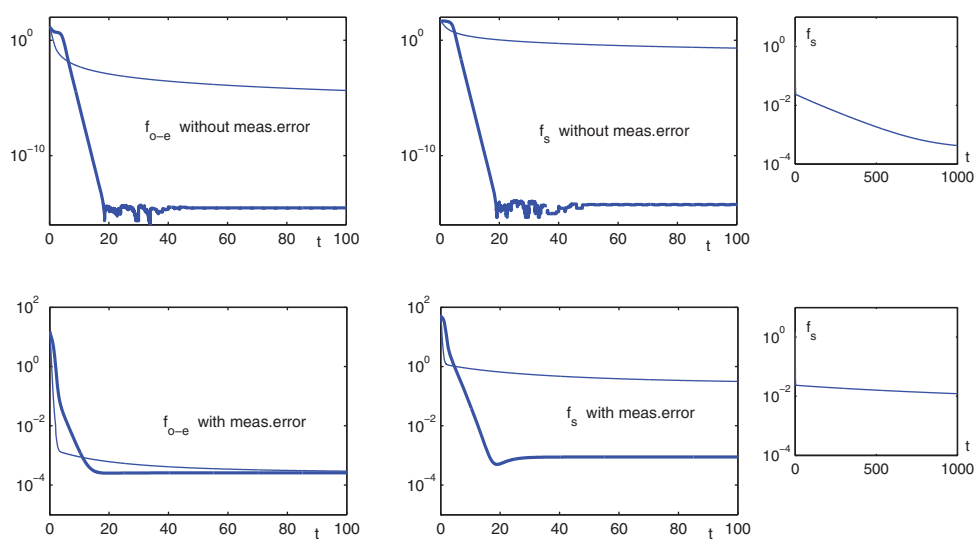

FIG. 3. *Evolution of* f_{o-e} *and* f_s *for six all-to-all connected agents, random initial conditions: thick lines generic* y_{ij} *(like case* (b) *of Theorem* 6*), thin lines* $y_{ij} = \frac{y_i - y_j}{\pm\|y_i - y_j\|_2}$ *with generic* y_i *(like case* (a) *of Theorem* 6*). Top plots without measurement errors, bottom plots with measurement errors. Far right plots: pursuing the simulation of case* (a) *to larger times.*

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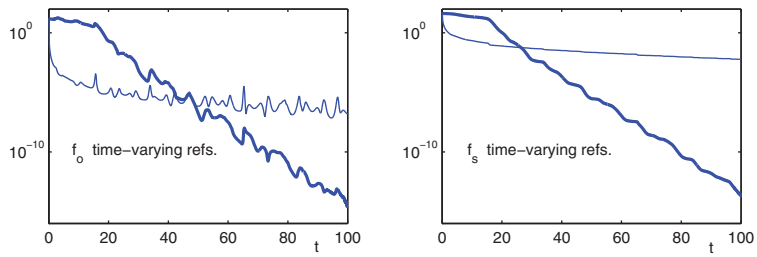

FIG. 4. *Evolution of f_o and f_s for six all-to-all connected agents, $y_{ij} = \frac{y_i - y_j}{\pm\|y_i - y_j\|_2}$ with time-varying $y_i(t)$. No measurement errors. Thick lines $\varepsilon = 0.1$ in* (16); *thin lines $\varepsilon = 10$.*

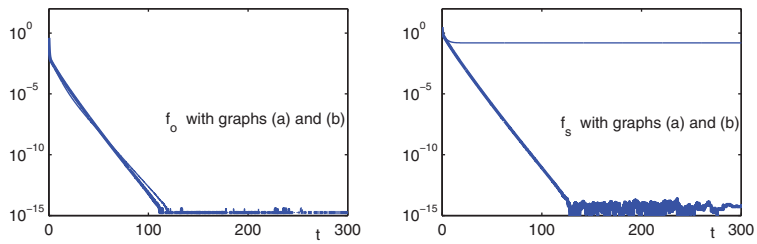

FIG. 5. *Evolution of f_o and f_s for seven agents with generic $y_{ij} \in S^2$ and initial conditions in the neighborhood of C_s, with interconnection graph of Figure* 2(a) *(thick lines) and of Figure* 2(b) *(thin lines). No measurement errors.*

are without measurement errors: for B, f_o and f_s exponentially decrease to machine precision; for A the decay is nonexponential. The conditions of Theorem 4 both fail for A, while they hold for B. Bottom plots repeat the simulation with random measurement errors $\|E_{ij} - I_3\|_F \leq \varepsilon = 0.01$. For both cases f_{o-e} stabilizes between 10^{-4} and 10^{-3}, indicating that the number of interconnections is so large that $C_{o-e} = \emptyset$ and f_{o-e} does not attain the value 0. Regarding f_s, case A now actually *saturates* at a significantly higher value than B. This is reminiscent of Theorem 7: for case (b) the synchronization error is of order ε; for case (a) it can be of order $\sqrt{\varepsilon} > \varepsilon$. The difference between subexponential convergence and saturation for A becomes apparent only on long simulations (small plots in Figure 3). The setting with y_{ij} given by relative positions seems ill-behaved in practice.

Figure 4 illustrates behavior with time-varying y_{ij}. Again we consider six all-to-all-coupled agents (no measurement errors). Reference vectors are derived from generic positions in $\mathbb{R}^3$, that is, $y_{ij} = \frac{y_i - y_j}{\pm\|y_i - y_j\|_2}$, where the $y_i(t)$ vary quasi-periodically in time with frequencies in $[0.05, \frac{1}{2\pi}]$ Hz. For $\varepsilon = 0.1$ in (16), roughly exponential convergence to $f_o = f_s = 0$ is observed (thick lines). Note that this ε is not too small w.r.t. frequencies of $y_i(t)$. The value $\varepsilon = 10$ (thin line) is too large for exploiting the time-varying setting: corresponding curves look more like the ones for fixed y_{ij} (thin lines of Figure 3).

Finally, we illustrate the influence of graph $\mathcal{G}$ on convergence. Figure 5 shows local evolution of f_o and f_s for seven agents interconnected according to graphs (a) and (b) of Figure 2. Both runs use the same generic fixed y_{ij} (except for the missing link in graph (b)) and same initial conditions in the neighborhood of C_s. Convergence to $f_o = 0$ is similar for both graphs. For graph (a) (thick lines), in accordance with Corollary 1, f_s also converges to 0. For graph (b) (thin lines), f_s converges to an

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arbitrary value, as $C_o \setminus C_s$ admits a continuum of relative states $(Q_2Q_1^T, \ldots, Q_7Q_1^T) \neq (I_3, \ldots, I_3)$.

6. Conclusion. This paper studies autonomous synchronization of agent orientations $Q_i \in SO(n)$ coupled through partial state information: a link between two agents allows them to compare the expression of a vector in their respective body frames. This setting associates an output map to each link, in contrast to previous studies on linear systems with an output map associated to each agent. A gradient law generalizing full state coupling [34] is proposed. Its convergence is studied in detail, with both necessary conditions and sufficient ones for state synchronization. An algorithm is proposed to find or build networks in which output synchronization implies state synchronization on $SO(3)$ for generic reference vectors. The setting with reference vectors derived as fixed relative positions in $\mathbb{R}^3$ is shown to be nonrobust. An analog system with states $x_i \in \mathbb{R}^n$ is studied for comparison.

We believe that this study can be both of practical interest—to design synchronizing networks with less than full state exchange—and of theoretical interest—illustrating nonrobust cases, highlighting further differences between $\mathbb{R}^n$ and $SO(n)$, and maybe motivating a more detailed study of orbit intersections on homogeneous spaces and Lie groups. The nontrivial issues raised by attributing output maps to *interactions* rather than to agents might motivate further study in the context of general distributed systems. One could study how this modified context affects, for example, swarm controllability to any joint state through an input at one of the agents [30] or conversely robustness to single agent perturbations [5].

7. Appendix.

LEMMA 1. *Consider an equilibrium* $\bar{Q} := (\bar{Q}_1, \ldots, \bar{Q}_k)$ *of* (6). *Define* $C_s^r := \{(X, \ldots, X) \in (\mathbb{R}^{n\times n})^k : X \in \mathbb{R}^{n\times n}\}$ *and* $F_{ij} = \bar{Q}_i^T M_{ij} \bar{Q}_j$. *Build the symmetric matrix* $L^{gg} = (L_{ij}^{gg}) \in \mathbb{R}^{kn\times kn}$ *with* $L_{ij}^{gg} = -a_{ij}F_{ij}$ *for* $i \neq j$ *and* $L_{ii}^{gg} = \sum_j a_{ij} F_{ij}$. *Then* $S = \{(\bar{Q}_1Q^*, \ldots, \bar{Q}_kQ^*) : Q^* \in SO(n)\}$ *is a set of equilibria, and* S *is locally exponentially stable if and only if* $(I_n \otimes L^{gg})$ *is positive definite on the subspace of* $\mathrm{vec}(\mathfrak{so}(n)^k)$ *orthogonal to* $\mathrm{vec}(C_s^r)$.

Nota Bene: The condition is necessary *only for* exponential *stability.*

Proof. At a critical point x the Hessian quadratic form $H_f(x)\colon (T_x\mathcal{M} \times T_x\mathcal{M}) \to \mathbb{R}$ of a smooth function $f\colon \mathcal{M} \to \mathbb{R}$ on a smooth manifold $\mathcal{M}$ is fully defined by: $H_f(x)(v, v) = \frac{d^2}{dt^2}(f \circ \gamma)(0)$ for any smooth curve $\gamma(t)$ on $\mathcal{M}$ with $\gamma(0) = x$ and $\frac{d}{dt}\gamma|_0 = v$; see [11]. On a Riemannian manifold with Riemannian metric $\langle\cdot,\cdot\rangle$, one can then define, on the tangent space at the critical point, a self-adjoint operator $\mathtt{H}_f(x)\colon T_x\mathcal{M} \to T_x\mathcal{M}$ such that $\langle v, \mathtt{H}_f(x)v\rangle = H_f(x)(v, v)$ for any $v \in T_x\mathcal{M}$. This $\mathtt{H}_f$ corresponds to linearization of the gradient vector field of f around the critical point x in exponential charts; therefore, $x \in \mathcal{M}$ is locally exponentially stable under gradient descent of f if and only if $H_f(x)$ is strictly positive definite.

Right-invariance of (6) implies that S is a set of equilibria on which $f_o = f_o(\bar{Q})$ is constant. Moreover, a curve with tangent vector in $\widetilde{Q}(\mathfrak{so}(n)^k \cap C_s^r) = \{(Q_1\Omega_b, \ldots, Q_k\Omega_b) : \Omega_b \in \mathfrak{so}(n)\}$ at each of its points $\widetilde{Q} = (Q_1, \ldots, Q_k) \in (SO(n))^k$ follows invariance directions of the dynamics. As in the proof of Theorem 5 we project the dynamics onto the reductive homogeneous space $\mathcal{M} = (SO(n))^k/C_s$ and denote by $\pi\colon (SO(n))^k \to \mathcal{M}$ the canonical projection. The function f_o is constant on the fibers of π and induces a smooth function $\hat{f}_o$ on $\mathcal{M}$ which satisfies $f_o = \hat{f}_o \circ \pi$. Since (6) is invariant under right multiplication of all Q_i by the same $Q \in SO(n)$, it induces a vector field X on $\mathcal{M}$. The set S is collapsed by π onto a single equilibrium $s \in \mathcal{M}$ and

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it is locally exponentially stable under (6) if and only if s is locally exponentially stable under X. We equip $\mathcal{M}$ with the induced normal Riemannian metric $\langle\cdot,\cdot\rangle_{\mathcal{M}}$ and denote by $\mathfrak{h}(\widetilde{Q})$, $\widetilde{Q} \in (SO(n))^k$, the canonical horizontal distribution on $(SO(n))^k$, i.e., the orthogonal complement of the vertical distribution $\mathfrak{v}(\widetilde{Q}) := \ker(T_{\widetilde{Q}}\pi) = \widetilde{Q}(\mathfrak{so}(n)^k \cap C_s^r)$. By construction, the vertical part $(\operatorname{grad} f_o(\widetilde{Q}))^{\mathfrak{v}} \in \mathfrak{v}(\widetilde{Q})$ of $\operatorname{grad} f_o(\widetilde{Q})$ is zero. Recall that for the normal metric, $\langle V, W\rangle_{(SO(n))^k} = \langle T_{\widetilde{Q}}\pi(V), T_{\widetilde{Q}}\pi(W)\rangle_{\mathcal{M}}$ for all $\widetilde{Q} \in (SO(n))^k$, $V, W \in \mathfrak{h}(\widetilde{Q})$ [9]. Some calculations show that the gradient system in $(SO(n))^k$ projects to the gradient system on $\mathcal{M}$: $X(\pi(\widetilde{Q})) = -\operatorname{grad}_{\mathcal{M}} \hat{f}_o(\pi(\widetilde{Q}))$. Then $s \in \mathcal{M}$ is locally exponentially stable if and only $H_{\hat{f}_o}(s)$ is strictly positive definite. Lifting curves from $\mathcal{M}$ to $SO(n)^k$ horizontally, $H_{\hat{f}_o}(s)$ is strictly positive definite if and only if $H_{f_o}(\bar{Q})$ is strictly positive definite on $\mathfrak{h}(\bar{Q})$. Take $\gamma\colon \mathbb{R} \to (SO(n))^k$ with $\gamma(0) = (\bar{Q}_1, \ldots, \bar{Q}_k)$ and $\frac{d}{dt}\gamma(0) = (\bar{Q}_1\Omega_1, \ldots, \bar{Q}_k\Omega_k) =: \bar{Q}\bar{\Omega} \in T_{\bar{Q}}(SO(n))^k$ with $\Omega_i \in \mathfrak{so}(n)$; also denote $\bar{\Omega} = (\Omega_1, \ldots, \Omega_k) \in \mathbb{R}^{n\times nk}$. From (5),

$$H_{f_o}(\bar{Q})(\bar{Q}\bar{\Omega}, \bar{Q}\bar{\Omega}) = \operatorname{tr}(\bar{\Omega}L^{gg}\bar{\Omega}^T) = (\operatorname{vec}(\bar{\Omega}^T))^T(I_n \otimes L^{gg})(\operatorname{vec}(\bar{\Omega}^T)).$$

Thus H_{f_o} is strictly positive definite on $\mathfrak{h}(\bar{Q})$ if and only if $(I_n \otimes L^{gg})$ is positive definite on the subspace of $\operatorname{vec}(\mathfrak{so}(n)^k)$ orthogonal to $\operatorname{vec}(C_s^r)$. ∎

The following result is used to prove Theorem 7. Let $h\colon SO(3) \to S^2$ given by $h(Q) = Q^T y$ for $y \in S^2$ and denote $h^{-1}(p) = \{Q : Q^T y = p\}$ for $p \in S^2$.

PROPOSITION 4. *If $y, z \in S^2$ are linearly independent, then there is a neighborhood $\mathcal{U}$ of I_3 in $SO(3)$ such that for all $p \in S^2$, $R \in \mathcal{U}$, $S \in SO(3)$, the set $h^{-1}(p) \cap R\operatorname{stab}(z)S$ contains at most one element.*

Proof. Note that $R\operatorname{stab}(z)R^T = \operatorname{stab}(Rz)$. One checks that $h^{-1}(p)S^TR^T = h^{-1}(RSp)$. Thus $\big(h^{-1}(p) \cap R\operatorname{stab}(z)S\big) = \big(h^{-1}(RSp) \cap \operatorname{stab}(Rz)\big)RS$. Given linearly independent z, y, there is a neighborhood $\mathcal{U}$ of I_3 in $SO(3)$ such that y and $v := Rz$ remain linearly independent for all $R \in \mathcal{U}$. This linear independence implies that equation $X^Ty = RSp$ has at most one solution $X \in \operatorname{stab}(v) = \operatorname{stab}(Rz)$ for all $p \in S^2$. ∎

Proof of Theorem 7. The characterization of C_{o-e} is invariant under right multiplication of the states by a common rotation; to simplify notation in this static context, we characterize C_{o-e} by first assuming $Q_1 = I_3$ and at the end reintroducing a common right multiplication. A (hypothetical) state (I_3, Q_2, Q_3) in C_{o-e} satisfies $Q_2E_{12}y_{12} = E_{21}y_{12}$, $Q_3E_{13}y_{13} = E_{31}y_{13}$, and $Q_3Q_2^TE_{23}y_{23} = E_{32}y_{23}$. The first two constraints yield $Q_2 \in E_{21}\operatorname{stab}(y_{12})E_{12}^T$ and $Q_3 \in E_{31}\operatorname{stab}(y_{13})E_{13}^T$. The output of agent 2 for y_{23} is then contained in the circle

$$Z_2' = \{E_{12}QE_{21}^TE_{23}y_{23} : Q \in \operatorname{stab}(y_{12})\} = \{y \in S^2 : y_{12}^TE_{12}^Ty = y_{12}^TE_{21}^TE_{23}y_{23}\}.$$

Similarly, output of agent 3 for y_{23} is contained in the circle

$$Z_3' = \{E_{13}QE_{31}^TE_{32}y_{23} : Q \in \operatorname{stab}(y_{13})\} = \{y \in S^2 : y_{13}^TE_{13}^Ty = y_{13}^TE_{31}^TE_{32}y_{23}\}.$$

For a state in C_{o-e}, these two outputs must be equal, so they must belong to $Z_2' \cap Z_3'$. Since the pairs y_{12}, y_{23} and y_{13}, y_{23} are each linearly independent, each point $p \in Z_2' \cap Z_3'$ corresponds to exactly one state of the form $(I_3, Q_2, Q_3) \in C_{o-e}$ for sufficiently small perturbations $\|E_{ij} - I_3\|_F$ by Proposition 4. (Existence of the state is ensured by definition of Z_2' and Z_3'.) Moreover, y_{12}, y_{13} linearly independent ensures that $Z_2' \cap Z_3'$ can contain at most two points for sufficiently small $\|E_{ij} - I_3\|_F$. Let

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us introduce $G_3 := E_{31}^T Q_3 E_{13}$, $G_2 := E_{13}^T E_{12} E_{21}^T Q_2 E_{13}$, $\bar{y}_{12} = E_{13}^T E_{12} y_{12}$, $\bar{y}_{23} = E_{31}^T E_{32} y_{23}$, and $E_A = E_{13}^T E_{12} E_{21}^T E_{23} E_{32}^T E_{31}$. A state being in C_{o-e} then requires $(G_2, G_3) \in \mathrm{stab}(\bar{y}_{12}) \times \mathrm{stab}(y_{13})$. Further, define

$$Z_2 := E_{13}^T Z_2' = \{y \in S^2 : \bar{y}_{12}^T y = \bar{y}_{12}^T (E_A \bar{y}_{23})\},$$
$$Z_3 := E_{13}^T Z_3' = \{y \in S^2 : y_{13}^T y = y_{13}^T \bar{y}_{23}\}.$$

Again, there is a bijection between points in $Z_2 \cap Z_3$ and states in C_{o-e} with $Q_1 = I_3$. We also define the balls

$$B_2 := \{y \in S^2 : \bar{y}_{12}^T y \geq \bar{y}_{12}^T (E_A \bar{y}_{23})\}, \quad B_3 := \{y \in S^2 : y_{13}^T y \geq y_{13}^T \bar{y}_{23}\},$$

whose boundaries are Z_2, respectively, Z_3. Furthermore, we denote by Z_{2o}, Z_{3o} the circles and B_{2o}, B_{3o} the balls for the unperturbed case, i.e., for $E_{ij} = I_3$.

From invariance of the Frobenius distance, $\|I_3 - J_2\|_F \leq \|I_3 - J_1\|_F + \|I_3 - J_1^T J_2\|_F$ for any $J_1, J_2 \in SO(3)$. This allows us to show that for $(E_{12}, \ldots, E_{32}) \in E_\varepsilon$,

$$\|Q - J_E\|_F \leq m\varepsilon \text{ if } J_E \text{ is the product of } Q \text{ with } m \text{ error rotations } E_{ij} \tag{21}$$

arbitrarily split into m_1 left and m_2 right multiplications, $m_1 + m_2 = m$. Also, $x_1^T (I_3 - J_1) x_2 \leq \|I_3 - J_1\|_F$ for $J_1 \in SO(n)$, $x_1, x_2 \in S^2$.

We restrict ourselves to the case $0 < y_{12}^T y_{13} < y_{12}^T y_{23} \leq y_{13}^T y_{23}$; the proof for other cases is strictly analogous. We redefine $\mathbf{e}_1 := \frac{\bar{y}_{12} \times y_{13}}{\|\bar{y}_{12} \times y_{13}\|_2}$ and $\hat{y}_{23} = \frac{(I_3 - \mathbf{e}_1 \mathbf{e}_1^T)\bar{y}_{23}}{\|(I_3 - \mathbf{e}_1 \mathbf{e}_1^T)\bar{y}_{23}\|_2}$.

(a) For Theorem 6(a), circles Z_{2o} and Z_{3o} intersect at a single tangency point y_{23}; then (in our restricted case, else balls must be redefined) the same is true for balls B_{2o} and B_{3o}. First consider the second assertion of the proposition. For any $\varepsilon > 0$ we can find a $\beta > 0$ and an open subset $N_\varepsilon \subset E_\varepsilon$ of $(SO(3))^6$ such that $y_{13}^T \bar{y}_{23} > y_{13}^T y_{23}$, $\bar{y}_{12}^T E_A \bar{y}_{23} > y_{12}^T y_{23} + \beta$, and $\bar{y}_{12}^T y_{12} \geq 1 - \frac{\beta}{2}$ for all perturbations $(E_{12}, \ldots, E_{32}) \in N_\varepsilon$. Then Z_2 and Z_3 are in the interior of B_{2o} and B_{3o}, respectively, so they cannot intersect and $C_{o-e} = \emptyset$.

Now consider the first assertion. For any $\varepsilon > 0$ sufficiently small, we can find an $\alpha > 0$ and an open subset $N_\varepsilon \subset E_\varepsilon$ of $(SO(3))^6$ such that for perturbations $(E_{12}, \ldots, E_{32}) \in N_\varepsilon$,

$$\begin{aligned}
&\text{(i)} && y_{13}^T \bar{y}_{23} < y_{13}^T E_A \bar{y}_{23} - \alpha\varepsilon, \\
&\text{(ii)} && \bar{y}_{12}^T \bar{y}_{23} > \bar{y}_{12}^T E_A \bar{y}_{23} + \alpha\varepsilon, \text{ and} \\
&\text{(iii)} && \|\bar{y}_{23} - \hat{y}_{23}\|_2 < \tfrac{\alpha}{2}\varepsilon\,.
\end{aligned}$$

From (i), $E_A \bar{y}_{23}$, on the boundary of B_2, also belongs to interior of B_3, so $B_2 \cap B_3$ has nonzero measure and $Z_2 \cap Z_3$ contains two points, corresponding to two separated sets in C_{o-e}. Take a state $(Q_1^*, Q_2^*, Q_3^*) \in C_{o-e}$ with corresponding (G_{2*}, G_{3*}) and $y_* = G_{3*} \bar{y}_{23} = G_{2*} E_A \bar{y}_{23}$. Then (ii) implies $\bar{y}_{12}^T y_* = \bar{y}_{12}^T G_{3*} \bar{y}_{23} = \bar{y}_{12}^T E_A \bar{y}_{23} < \bar{y}_{12}^T \bar{y}_{23} - \alpha\varepsilon$ so $\bar{y}_{12}^T (I_3 - G_{3*}) \bar{y}_{23} > \alpha\varepsilon$ and using (iii) we get $\bar{y}_{12}^T (I_3 - G_{3*}) \hat{y}_{23} > \frac{\alpha}{2}\varepsilon$. Writing G_{3*} and vectors in orthonormal basis $(\mathbf{e}_1, y_{13}, \mathbf{e}_1 \times y_{13})$ shows that this requires G_{3*} to be a rotation of angle $\theta(\varepsilon)$ with $(1 - \cos(\theta(\varepsilon))) > c_1 \varepsilon$ for some constant c_1 independent of ε; then there is a constant $c_2 > 0$ independent of ε such that $\theta(\varepsilon) > c_2 \sqrt{\varepsilon}$. Thus we can find, for ε sufficiently small, a constant $c_3 > 0$ independent of ε such that

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$\|I_3 - G_{3*}\|_F = \sqrt{8}\sin(\frac{\theta(\varepsilon)}{2}) > c_3\sqrt{\varepsilon}$. Using property (21) and

$$\begin{aligned}
\operatorname{dist}(C_s, C_{o-e}) &= \min_{\substack{Q\in SO(n)\\(I_3,Q_2,Q_3)\in C_{o-e}}} \left(\|Q-I_3\|_F^2 + \|Q-Q_2\|_F^2 + \|Q-Q_3\|_F^2\right)^{1/2} \\
&\geq \frac{1}{\sqrt{3}} \min_{\substack{Q\in SO(n)\\(I_3,Q_2,Q_3)\in C_{o-e}}} \|Q-I_3\|_F + \|Q-Q_2\|_F + \|Q-Q_3\|_F \\
&\geq \frac{1}{\sqrt{3}} \min_{\substack{Q\in SO(n)\\(I_3,Q_2,Q_3)\in C_{o-e}}} \|Q-I_3\|_F + \|Q-Q_3\|_F \\
&\geq \frac{1}{\sqrt{3}} \min_{(I_3,Q_2,Q_3)\in C_{o-e}} \|I_3-Q_3\|_F
\end{aligned}$$

we get, for sufficiently small ε, a constant $c_4 > 0$ independent of ε such that

$$\begin{aligned}
\operatorname{dist}(C_s, C_{o-e}) \geq \tfrac{1}{\sqrt{3}}\|I_3 - Q_{3*}\|_F &\geq \frac{\|I_3 - G_{3*}\|_F - \|G_{3*} - Q_{3*}\|_F}{\sqrt{3}} \\
&\geq \frac{1}{\sqrt{3}}\|I_3 - G_{3*}\|_F - 2\varepsilon \geq c_4\sqrt{\varepsilon}\,.
\end{aligned}$$

(b) Denote by θ the angle between y_{23} and y_{12}, i.e., $y_{12}^T y_{23} = \cos(\theta)$, with $\theta \in (0, \frac{\pi}{2}]$ since we consider $y_{12}^T y_{23} > 0$. Define $B_\alpha = \{y \in S^2 : \|y - y_{23}\|_2 \leq \alpha\}$. Theorem 6(b) requires Z_{2o} and Z_{3o} to intersect at two points. This requires their intersections to be transversal. As a consequence, there exist $\beta_0, \gamma_0 > 0$ such that $\alpha < \gamma_0$ implies (i) following the boundary of B_α, one passes through four points $p_{2a}, p_{3a}, p_{2b}, p_{3b}$ in that order with $\{p_{2a}, p_{2b}\} \subset Z_{2o}$ and $\{p_{3a}, p_{3b}\} \subset Z_{3o}$; and (ii) on each boundary arc $[p_i, p_j]$, there is at least one point $r_{[p_i,p_j]}$ whose distances to Z_{2o} and Z_{3o} are both strictly larger than $\beta_0\alpha$. Next, note that there can be no point belonging to Z_2 (resp., Z_3) whose distance to Z_{2o} (resp., to Z_{3o}) is larger than $\frac{4\varepsilon}{\sin(\theta/2)}$. Indeed, denote $x \in S^2$ a point at distance δ from Z_{2o} in $\mathbb{R}^3$, and denote $\phi \in [0, \pi]$ the angle between vectors x and y_{12}; then $|\cos(\theta) - \cos(\phi)| \geq \delta\sin(\frac{\theta+\phi}{2}) \geq \delta\sin(\frac{\theta}{2})$. At the same time, $x \in Z_2$ requires $\bar{y}_{12}^T(x - E_A\bar{y}_{23}) = 0$, hence

$$\begin{aligned}
|\cos(\theta) - \cos(\phi)| &= |y_{12}^T(x - y_{23}) \;-\; \bar{y}_{12}^T(x - E_A\bar{y}_{23})| \\
&= |y_{12}^T(I_3 - E_{12}^T E_{13})x + y_{12}^T(I_3 - E_{21}^T E_{23})y_{23}| \\
&\leq \|y_{12}\|_2(\|I_3 - E_{12}^T E_{13}\|_F\|x\|_2 + \|I_3 - E_{21}^T E_{23}\|_F\|y_{23}\|_2) \;\leq\; 4\varepsilon.
\end{aligned}$$

These two inequalities yield $\delta < 4\varepsilon/\sin(\frac{\theta}{2})$. A similar argument applies for Z_3.

As Z_2 and Z_3 are thus confined to neighborhoods of Z_{2o} and Z_{3o}, take $\alpha < \gamma_0$ and $\varepsilon = \frac{\beta_0\alpha\sin(\theta/2)}{4}$. Then the intersection points of Z_2 and Z_3 with the boundary of B_α must remain confined in their respective arcs delimited by the $r_{[p_i,p_j]}$, e.g., Z_2 has one intersection with the boundary of B_α on the arc between $r_{[p_{2a},p_{3a}]}$ and $r_{[p_{3b},p_{2a}]}$, and similarly for the others; thus the order of intersections along the boundary—alternating between Z_2 and Z_3—is conserved. This implies that Z_2 and Z_3 have an intersection point $y_* \in B_\alpha$, characterizing a state $(I_3, Q_{2*}, Q_{3*}) \in C_{o-e}$ associated to $(G_{2*}, G_{3*}) \in \operatorname{stab}(\bar{y}_{12}) \times \operatorname{stab}(y_{13})$ and $y_* = G_{3*}\bar{y}_{23} = G_{2*}E_A\bar{y}_{23}$. Then using

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property (21),

$$\begin{aligned}\operatorname{dist}(C_s, C_{o-e}) &= \min_{\substack{Q\in SO(n)\\(I_3,Q_2,Q_3)\in C_{o-e}}} \left(\|Q-I_3\|_F^2 + \|Q-Q_2\|_F^2 + \|Q-Q_3\|_F^2\right)^{1/2}\\ &\le \min_{\substack{Q\in SO(n)\\(I_3,Q_2,Q_3)\in C_{o-e}}} \|Q-I_3\|_F + \|Q-Q_2\|_F + \|Q-Q_3\|_F\\ &\le \|I_3-Q_{2*}\|_F + \|I_3-Q_{3*}\|_F \;\le\; \|I_3-G_{2*}\|_F + \|I_3-G_{3*}\|_F + 6\varepsilon\,.\end{aligned}$$

By the choice of α and $B_\alpha \ni y_*$, we have $\|y_*-y_{23}\|_2 \le \frac{4\varepsilon}{\beta_0\sin(\theta/2)}$. Moreover, as G_{3*} is a rotation around y_{13}, we have $\|I_3-G_{3*}\|_F = \frac{\sqrt{2}}{r_{3*}}\|y_*-\bar{y}_{23}\|_2$ with $r_{3*} = \|y_{13}\times\bar{y}_{23}\|_2$; similarly $\|I_3-G_{2*}\|_F = \frac{\sqrt{2}}{r_{2*}}\|y_*-E_A\bar{y}_{23}\|_2$ with $r_{2*} = \|\bar{y}_{12}\times(E_A\bar{y}_{23})\|_2$. There exist $\beta_1,\gamma_1>0$ such that $\varepsilon<\gamma_1$ implies $\min(\|\bar{y}_{12}\times(E_A\bar{y}_{23})\|_2,\ \|y_{13}\times\bar{y}_{23}\|_2) > \sqrt{2}\beta_1$. Then

$$\begin{aligned}\operatorname{dist}(C_s, C_{o-e}) &\le \frac{\|y_*-\bar{y}_{23}\|_2 + \|y_*-E_A\bar{y}_{23}\|_2}{\beta_1} + 6\varepsilon\\ &\le \frac{2\|y_*-y_{23}\|_2 + 6\varepsilon}{\beta_1} + 6\varepsilon \;\le\; \alpha_2\varepsilon\end{aligned}$$

for small enough ε with $\alpha_2 = 6 + \frac{1}{\beta_1}\left(6 + \frac{8}{\beta_0\,\sin(\theta/2)}\right)$ finite. ∎

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