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Synchronization with partial state feedback on $SO(n)$

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***Abstract*— In this paper we consider the problem of constructing a distributed feedback law to achieve synchronization for a group of k agents whose states evolve on $SO(n)$ and which exchange only partial state information along communication links. The partial state information is given by the action of the state on reference vectors in $\mathbb{R}^n$. We propose a gradient based control law which achieves exponential local convergence to a synchronization configuration under a rank condition on a generalized Laplacian matrix. Furthermore, we discuss the case of time-varying reference vectors and provide a convergence result for this case. The latter helps reach synchronization, requiring less communication links and weaker conditions on the instantaneous reference vectors. Our methods are illustrated on an attitude synchronization problem where agents exchange only their relative positions observed in the respective body frames.**

I. Introduction

Recently, synchronization and other collective phenomena, appearing in physical and other natural systems, have drawn considerable attention in the literature, see e.g. [22], [26], [19]. The systems and control community has been studying coordinated multi-agent systems; potential capabilities indicate that they may be increasingly used in future applications involving e.g. communication networks or vehicle formations, see e.g. [15], [23], [8], [9], [13], [10], [14], [5], [7]. Considered aspects include optimal configuration of a group, collision avoidance, nonlinear dynamics, communication graph structures, distributed controller design, etc.

One main line of research derives from the standard "linear consensus algorithm", used by a set of interacting agents to reach agreement on some vector in $\mathbb{R}^n$, see e.g. [23], [9], [16]. Problems like rigid body attitude synchronization motivate an extension of the "linear consensus algorithm" to manifolds different from $\mathbb{R}^n$; indeed, satellite attitudes for instance evolve on the group $SO(3)$ of rotation matrices. The attitude synchronization problem has already been studied in e.g. [15], [25], [12], [3], [11], [18], [21] with different approaches. All these studies consider full state exchange between communicating agents. A setting with partial state feedback in *linear* systems is proposed and analyzed in [24].

The present paper extends a basic control law for synchronization on $SO(n)$ (see [20]) to a setting where agents only exchange partial state information. Our output map is inspired by [17], which takes advantage of the fact that it comes down to full state communication for $SO(2)$ in order to achieve smart noise reduction. We design a gradient algorithm on the basis of a cost function on the outputs. Considering a static output map, we provide a sufficient rank condition on a generalized graph Laplacian matrix which ensures that the resulting system reaches state synchronization. Then we consider a time-varying output map; a sufficient condition is provided to get local state synchronization under appropriate persistance of excitation assumptions. The time-varying setting leads to conditions that are easier to satisfy, because only a local rank condition must hold for the *averaged* system; this implies that synchronization can be ensured with less communication links among agents.

The paper is organized as follows. Section II states the problem and derives the basic gradient algorithm used as control law by the individual agents. Section III analyzes the stability properties for fixed output maps. Section IV considers time-varying output maps. Section V illustrates the results on an example with simulations.

II. Problem statement and gradient synchronization algorithm

A. Definitions and Notation

We denote by $SO(n)$ the set of n-dimensional rotation matrices, that is the Lie group of orthogonal $n \times n$ matrices with determinant 1, and by $\mathfrak{so}(n)$ the tangent space to $SO(n)$ at identity, that is the Lie algebra of $n \times n$ skew symmetric matrices. We equip $SO(n)$ with the biinvariant Riemannian metric $\langle Q\Omega, Q\Theta\rangle = \mathrm{tr}(\Omega^T\Theta)$ for $Q \in SO(n)$, $\Omega, \Theta \in \mathfrak{so}(n)$. The Frobenius norm, which coincides with the induced norm on the tangent spaces of $SO(n)$, is denoted by $\|\cdot\|$. For a matrix $X \in \mathbb{R}^{n\times n}$, $\mathrm{skew}(X)$ denotes its skew-symmetric part, i.e. $\frac{1}{2}(X - X^T)$, and $\mathrm{vec}(X)$ the map $\mathbb{R}^{n\times n} \to \mathbb{R}^{n^2}$ such that $\mathrm{vec}((x_{ij})) = (x_{11}, x_{21}, \ldots x_{n1}, x_{12}, \ldots x_{n2}, \ldots, x_{1n}, \ldots x_{nn})^T$. The Kronecker product of two matrices A, B is denoted by $A \otimes B$. The unit sphere in $\mathbb{R}^n$ is denoted by S^{n-1}.

Inter-agent communication is represented by means of a communication graph (V, E), V denoting the vertices, i.e. the agents, and E the edges, i.e. the available communication links. Here communication links are assumed bidirectional, so the graph is undirected. The cardinality of E is denoted $\#E$. The adjacency matrix $A = (a_{ij})$ of (V, E) contains $a_{ij} = 1$ if there is a link between agents i and j, and $a_{ij} = 0$ otherwise. By convention $a_{ii} = 0$. Both vertices and edges are represented by numbers, i.e. $V = \{1, \ldots, k\}$ and $E = \{1, \ldots, \#E\}$. The vertices i, j linked by edge $e \in E$ are denoted $i = \mathrm{vl}(e) \in V$ and $j = \mathrm{vr}(e) \in V$, with $i < j$. The set of edges attached to a vertex i is denoted by $\mathrm{ed}(i)$. The degree of vertex i is denoted $\deg(i)$.

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B. Synchronization algorithm

Consider k agents whose state space is $SO(n)$ — e.g. k satellites with attitudes represented by $SO(3)$. We denote the state of agent i by $Q_i \in SO(n)$. We assume that the agents have the simple left-invariant dynamics

$$\tfrac{d}{dt}Q_i = Q_i u_i \quad , \quad i = 1 \dots k \tag{1}$$

with $u_i \in \mathfrak{so}(n)$ a freely chosen input. This yields a left-invariant system on the product group $SO(n)^k$. The agents communicate along links defined by a fixed undirected graph (V, E). However, as a major distinction with respect to previous work, if two agents i and j are connected by a communication link, i.e. $a_{ij} \neq 0$, then they only exchange the *partial* state information

$$Q_i^T y_{ij} \;\text{ and }\; Q_j^T y_{ij}$$

along that link. The $y_{ij} \in S^{n-1}$ are imposed reference vectors; we assume $y_{ij} = y_{ji}$, i.e. one vector is associated to each *bidirectional* link. This kind of output naturally appears in applications as e.g. described in Section V. We first consider fixed y_{ij}, then analyze how the system behaves for time-varying y_{ij}. The output map is given by $H\colon SO(n)^k \to (S^{n-1})^{2\#E}$ where, choosing a suitable order of the outputs, components j and $j + \#E$ of H, $j \in E$, are given by

$$\begin{aligned} r_j := H_j(Q_1, \dots, Q_k) &= Q^T_{\mathrm{vl}(j)} y_{\mathrm{vl}(j)\,\mathrm{vr}(j)} \\ s_j := H_{j+\#E}(Q_1, \dots, Q_k) &= Q^T_{\mathrm{vr}(j)} y_{\mathrm{vl}(j)\,\mathrm{vr}(j)}. \end{aligned}$$

The goal is to find a feedback law, defining u_i as a function of $\{(Q_i^T y_{ij},\, Q_j^T y_{ij}),\, j = 1 \dots k \mid a_{ij} \neq 0\}$, that drives the system to *synchronization* (or *state synchronization*), i.e. to the set $C_s = \{(Q_1, \dots, Q_k) \in SO(n)^k \mid Q_1 = \dots = Q_k\}$.

In [20] a gradient algorithm is proposed for (at least local) synchronization on $SO(n)$ with full state observations. Here, we extend this approach to partial state observations. For this we build a cost function $\hat{f}\colon (S^{n-1})^{2\#E} \to \mathbb{R}$ on *output* space that penalizes the difference between pairs of corresponding outputs r_j and s_j along each edge j. A natural choice is the sum of squared Euclidean distances

$$\hat{f}(r_1, \dots, r_{\#E}, s_1, \dots, s_{\#E}) = \sum_{j=1}^{\#E} \frac{1}{2} \|r_j - s_j\|^2.$$

The cost function on outputs can be pulled back via the output map to a cost function on state space $f = \hat{f} \circ H$,

$$\begin{aligned} f(Q_1, \dots, Q_k) &= \sum_{i=1}^{k} \sum_{j<i} a_{ij} \frac{1}{2} \|Q_i^T y_{ij} - Q_j^T y_{ij}\|^2 \\ &= \sum_{i=1}^{k} \sum_{j<i} a_{ij} (1 - \mathrm{tr}(Q_j^T y_{ij} y_{ij}^T Q_i)). \end{aligned}$$

We define $M_{ij} := y_{ij} y_{ij}^T$. Using the product Riemannian metric on $SO(n)^k$, the gradient of f with respect to Q_i is

$$\Big(\mathrm{grad}\, f(Q_1, \dots, Q_k)\Big)_i = -\sum_{j \neq i} a_{ij}\, Q_i\, \mathrm{skew}(Q_i^T M_{ij} Q_j). \tag{2}$$

As in the full state observation case, we use gradient dynamics as a synchronization algorithm: for $i = 1 \dots k$,

$$\begin{aligned} \tfrac{d}{dt}Q_i &= -\Big(\mathrm{grad}\, f(Q_1, \dots, Q_k)\Big)_i \\ &= Q_i \left(\sum_{j=1}^{k} a_{ij}\, \mathrm{skew}((Q_i^T y_{ij})(Q_j^T y_{ij})^T) \right). \end{aligned} \tag{3}$$

Law (3) is indeed an output feedback, depending only on the locally available partial state information.

III. Fixed reference vectors

We now discuss the convergence to synchronization of system (3) for *fixed* reference vectors y_{ij}. Since f characterizes output disagreement, we first consider stability of output synchronization, i.e. the situation for which the two outputs along each edge of (V, E) coincide.

Definition 1: A state $(Q_1, \dots, Q_k)$ is called *output synchronization* if $Q_i^T y_{ij} = Q_j^T y_{ij}$ for all $i, j \in \{1, \dots, k\}$ with $a_{ij} = 1$. We denote the set of all output synchronization states by C_o.

Proposition 1: The output synchronization set is asymptotically stable under (3).

Proof: Since (3) is a gradient descent system for f and f is an algebraic function the set C_o, i.e. the set of global minima of f, is asymptotically stable [1]. ∎

The next step is to examine when output synchronization implies state synchronization. We first consider a condition for outputs of the system to correspond to a unique state.

Theorem 1: The output map from the global state space to the output space is injective if and only if each $i \in V$ has at least $n-1$ linearly independent y_{ij} with $a_{ij} = 1$. If this condition does not hold, then the outputs corresponding to any state in C_o can be obtained with a state outside C_s.

Proof: Consider a single agent i. W.l.o.g. we choose the numbering of the agents and m such that $a_{ij} = 1$ if and only if $j \leq m$. Output $Q_i^T y_{ij}$ of agent i is given by the action of state Q_i on y_{ij}, $\forall j \leq m$. Let $h\colon SO(n) \times (S^{n-1})^m \to (S^{n-1})^m$ the output map action of Q_i on $\hat{y} = (y_{i1}, \dots, y_{im})$, i.e. $h(Q_i, \hat{y}) = (Q_i^T y_{i1}, \dots, Q_i^T y_{im})$. The stabilizer $\mathrm{stab}_h(\hat{y}) = \{Q \in SO(n) \mid h(Q, \hat{y}) = \hat{y}\}$ is

$$\mathrm{stab}_h(\hat{y}) = \bigcap_{j=1}^{m} \mathrm{stab}(y_{ij}),$$

with $\mathrm{stab}(y_{ij}) = \{Q \in SO(n) \mid Q^T y_{ij} = y_{ij}\}$. Thus by definition the $y_{i1}, \dots, y_{im}$ are eigenvectors with eigenvalue 1 of all $Q \in \mathrm{stab}_h(\hat{y}) \subseteq SO(n)$. Then $\mathrm{stab}_h(\hat{y}) = \{I\}$ if and only if $(n-1)$ elements of $\mathcal{Y}_i = \{y_{i1}, \dots, y_{im}\}$ are linearly independent. The same holds if $\hat{y}$ is replaced by $h(Q^*, \hat{y})$ for any $Q^* \in SO(n)$. Therefore an output of the whole system corresponds to a unique state of agent i if and only if there are $(n-1)$ linearly independent elements in $\mathcal{Y}_i$. We get injectivity of the whole output map if and only if this condition holds for all $i \in V$.

Now assume that the condition does not hold for agent i. Consider output synchronization with the synchronized state $Q_1 = \dots = Q_k$. For any $\hat{Q} \in \mathrm{stab}(\hat{y})$, the state $(Q_1, \dots, \hat{Q}Q_i, \dots, Q_k)$ yields the same output as $(Q_1, \dots, Q_k) \in C_s$. But as $\operatorname{stab}(\hat{y}) \neq \{I_n\}$, we can choose $\hat{Q} \neq I$, such that $(Q_1, \dots, \hat{Q}Q_i, \dots, Q_k) \notin C_s$. Hence there is no output value in C_o which necessarily requires the state to be in C_s. ■

Even if we have enough outputs to ensure injectivity of the output map, this still does not guarantuee that an output synchronization state is a state synchronization state. The problem is that the output values generated by the state synchronization set are only part of the possible output synchronization values: there exist output values $Q_i^T y_{ij} = Q_j^T y_{ij}$ that cannot be generated by synchronization states, as illustrated in Example 1. Thus depending on the actual *values* of the outputs, output synchronization can correspond to state synchronization or not. Therefore the above proposition only gives a necessary, but not a sufficient condition for equivalence of output and state synchronization.

Example 1: Consider the case of $SO(3)$, with 3 agents and a fully connected communication graph. Assume that $y_{12} = (1,0,0)^T$, $y_{23} = (0,1,0)^T$, $y_{13} = (0,0,1)^T$. The assumptions of Theorem 1 hold, so each output synchronization value corresponds to a unique point in state space. Let output synchronization be reached at $Q_1^T y_{12} = Q_2^T y_{12} = \begin{pmatrix}1 & 0 & 0\end{pmatrix}^T$, $Q_1^T y_{13} = Q_3^T y_{13} = \begin{pmatrix}0 & 0 & 1\end{pmatrix}^T$ and $Q_2^T y_{23} = Q_3^T y_{23} = \begin{pmatrix}0 & -1 & 0\end{pmatrix}^T$. Then the corresponding unique state is $Q_1 = \operatorname{diag}(1,1,1)$, $Q_2 = \operatorname{diag}(1,-1,-1)$, $Q_3 = \operatorname{diag}(-1,-1,1)$ which is not a state synchronization state.

To characterize when output synchronization yields state synchronization, we must take a closer look at cost function f. The function can be written as $f(Q_1, \dots, Q_k) =$

$$\frac{1}{2}\left(\operatorname{vec}\begin{pmatrix}Q_1\\ \vdots \\ Q_k\end{pmatrix}\right)^T (I_n \otimes L) \quad \operatorname{vec}\begin{pmatrix}Q_1\\ \vdots \\ Q_k\end{pmatrix}$$

where $L = (L_{ij}) \in \mathbb{R}^{kn \times kn}$, with $L_{ij} \in \mathbb{R}^{n \times n}$ defined by

$$\begin{aligned} L_{ij} &= -a_{ij} M_{ij} \quad \text{for } i \neq j\,, \\ L_{ii} &= \sum_{j=1}^{k} a_{ij} M_{ij}\,. \end{aligned}$$

L and f can be considered as generalizations of the Laplacian matrix and Laplacian-based quadratic cost functions often used in the context of synchronization algorithms.

Since $\sum_{j=1}^{k} L_{ij} = 0 \ \forall i$, we have $\operatorname{rank} L \leq n(k-1)$. If this bound is tight we can prove strong convergence properties of our gradient system.

Theorem 2: If $\operatorname{rank} L = n(k-1)$, then
(a) output synchronization C_o = state synchronization C_s;
(b) $C_o = C_s$ is the set of stable equilibria of (3);
(c) $C_o = C_s$ is locally exponentially stable.

Proof: (a) Consider $f(X_1, \dots, X_k)$ with $X_i \in \mathbb{R}^{n \times n}$. $C_o \subset S$ where S is the subspace of global minima of the quadratic form f on $(\mathbb{R}^{n \times n})^k$. In addition, $S \supseteq V$ with V the subspace $V := \{(X_1, \dots, X_k) \in (\mathbb{R}^{n \times n})^k \mid X_1 = \dots = X_k \in \mathbb{R}^{n \times n}\}$. If $\operatorname{rank} L = n(k-1)$, then $\operatorname{rank} I_n \otimes L = n^2 k - n^2$ and S is n^2-dimensional, like V, thus $S = V$. Then $C_o \subset S = V$ and taking the intersection with $SO(n)^k$ leads to $C_o = C_s$.

(b) Since f and $SO(n)$ are analytic, an equilibrium of gradient system (3) is stable if and only if it is a local minimum of f [1]. Similarly to [20], the fact that local minima are global ones for linear functionals on $SO(n)$ implies that all local minima of f on $SO(n)^k$ are global ones and hence belong to C_o.

(c) Recall that *at a critical point* x the Hessian $Hf(x)\colon (T_x M \times T_x M) \to \mathbb{R}$ of smooth function $f\colon M \to \mathbb{R}$ on a smooth manifold M can be defined by $Hf(x)(\eta,\eta) := \frac{d^2}{dt}(f \circ \gamma)(0)$, where $\gamma(t)$ is a smooth curve on M with $\gamma(0) = x$ and $\frac{d}{dt}\gamma|_0 = \eta$, cf. [6].

The Hessian $Hf(Q)$ of f on $SO(n)^k$ is positive semidefinite in $C_o = C_s$. We want to show that, for all $Q = (Q_1, \dots, Q_k) \in C_s$, $Hf(Q)(\eta,\eta) = 0$ implies $\eta \in T_Q C_s$. Since C_s is a compact submanifold of $SO(n)^k$, exponential stability then follows by the same argument as for Morse-Bott functions, cf. [6].

Computing H at a minimum $\gamma(0) = (\hat{Q}, \dots, \hat{Q}) \in C_s$ of f on $M = SO(n)^k$ with $\gamma\colon \mathbb{R} \to SO(n)^k$ a smooth curve and $\frac{d}{dt}\gamma(0) = \gamma(0)(\Omega_1, \dots, \Omega_k) = (\hat{Q}\Omega_1, \dots, \hat{Q}\Omega_k)$ where $\Omega_i \in \mathfrak{so}(n)$, we get

$$\frac{d^2}{dt^2}(f \circ \gamma)(0) = \left(\operatorname{vec}\begin{pmatrix}\Omega_1\\ \vdots \\ \Omega_k\end{pmatrix}\right)^T (I_{nk} \otimes \hat{Q}^T)(I_n \otimes L)(I_{nk} \otimes \hat{Q}) \ \operatorname{vec}\begin{pmatrix}\Omega_1\\ \vdots \\ \Omega_k\end{pmatrix}.$$

As discussed for case (a) we have for $\operatorname{rank} L = k(n-1)$ that $\ker(I_n \otimes L) = \operatorname{vec}(V)$ with V as above and $\operatorname{vec}(V)$ the image of V under the vectorization map vec. Defining $U := I_{nk} \otimes \hat{Q}$, we have $U \operatorname{vec}(V) = \operatorname{vec}(V)$ and since U is an orthogonal transformation, $\operatorname{rank} U^T (I_n \otimes L) U = \operatorname{rank}(I_n \otimes L)$. Therefore $\frac{d^2}{dt^2}(f \circ \gamma)|_0 = 0$ implies $(\Omega_1, \dots, \Omega_k) \in V \cap T_Q SO(n)^k = T_Q C_s$. ■

Obviously, synchronization can only be achieved for connected communication graphs. For a disconnected graph, numbering the agents in order to make L block-diagonal, one readily shows that $\operatorname{rank} L < n(k-1)$.

The maximal $\operatorname{rank} L = n(k-1)$ can only be achieved for a suitably large number of agents.

Proposition 2: If $\operatorname{rank} L = n(k-1)$ then $k \geq 2n$.

Proof: Define the vectors

$$\tilde{y}_{ij} = (\dots, 0, y_{ij}^T, 0, \dots, 0, -y_{ij}^T, 0, \dots)^T \in \mathbb{R}^{kn}$$

where the y_{ij} entries appear as ith and jth $\mathbb{R}^n$-components of $\tilde{y}_{ij}$. Further define the $kn \times k(k-1)/2$ matrix

$$P = \begin{pmatrix} a_{12}\tilde{y}_{12} & a_{13}\tilde{y}_{13} & \cdots & a_{23}\tilde{y}_{23} & a_{24}\tilde{y}_{24} & \dots \end{pmatrix} \quad (4)$$

whose columns are given by the $a_{ij}\tilde{y}_{ij}$ for $i < j$, sorted by (i,j) in lexicographic order. Then $L = PP^T$ and $\operatorname{rank} L = \operatorname{rank} P \leq \min\{n(k-1), k(k-1)/2\}$. Therefore $\operatorname{rank} L = (k-1)n$ requires $n(k-1) \leq k(k-1)/2$, i.e. $2n \leq k$. ■

It is important to note that Theorem 2 provides a *sufficient* condition for state synchronization. Thus Proposition 2 is *not* a necessary condition for state synchronization. For instance, state synchronization on $SO(3)$ can be ensured with $k = 4$ fully connected agents and suitably chosen y_{ij}, although Proposition 2 is not satisfied.

IV. TIME-VARYING REFERENCE VECTORS

In applications one can be faced with a situation where the reference vectors y_{ij} time-varying, i.e. the y_{ij} are smooth functions $\mathbb{R} \to S^{n-1}$. This setting is in fact favorable for state synchronization, as the output map can sweep in time different directions of the state space.

For technical reasons, we introduce a scaling parameter ε in (3), i.e. we consider for $i = 1 \dots k$ the system

$$\tfrac{d}{dt} Q_i = \varepsilon Q_i \left(\sum_{j=1}^{k} a_{ij} \operatorname{skew}((Q_i^T y_{ij})(Q_j^T y_{ij})^T) \right) \quad (5)$$

with time-varying $y_{ij}(t)$. Feedback law (5) can be construed as the gradient, with respect to the state space variables, of

$$f(Q_1, \dots, Q_k, t) = \sum_{i=1}^{k} \sum_{j<i} a_{ij} \, \frac{1}{2} \| Q_i^T y_{ij}(t) - Q_j^T y_{ij}(t) \|^2.$$

However, because this cost function explicitly depends on time, it does not necessarily decrease along the trajectories of the closed loop system and thus is not a Lyapunov function.

Using an averaging approach, we show that C_s is locally asymptotically stable for this time-varying gradient feedback if a persistent excitation condition on the outputs holds:

Assumption 1: For all $i, j = 1 \dots k$, $i \neq j$,

$$\overline{M}_{ij} := \lim_{t \to \infty} \frac{1}{2t} \int_{-t}^{t} y_{ij}(s) y_{ij}(s)^T ds$$

exists and is strictly positive definite.

Assumption 1 ensures that the system given by the average agent dynamics (w.l.o.g. for $\varepsilon = 1$)

$$\tfrac{d}{dt} \overline{Q}_i = \overline{Q}_i \left(\sum_{j=1}^{k} a_{ij} \operatorname{skew}(\overline{Q}_i^T \overline{M}_{ij} \overline{Q}_j) \right) \quad (6)$$

exists and that the whole relative-state space (i.e. all dimensions of the $Q_i^T Q_j$ variable, $\forall\, i, j$) is actually observed by integrating the output maps. The averaged system has the same convergence properties as the gradient system with full state observations introduced in [20].

Proposition 3: If the communication graph is connected, then the state synchronization set C_s is exponentially stable for the averaged system (6).

Proof: System (6) is a gradient system for the averaged cost function (note that the Q_i do not depend on s)

$$\overline{f}(Q_1, \dots, Q_k) = \lim_{t \to \infty} \frac{1}{2t} \int_{-t}^{t} f(Q_1, \dots, Q_k, s) ds = \sum_{i,j=1}^{k} a_{ij} \, \| \overline{M}_{ij} (Q_i - Q_j) \|^2.$$

Using a straightforward extension of the arguments in [20] we see that C_s is locally asymptotically stable. Exponential stability can be proved similarly to Theorem 2(c). ■

We can now derive the exponential stability of the time-varying system from the averaged one.

Theorem 3: If the communication graph is connected and Assumption 1 holds, then for sufficiently small $\varepsilon > 0$ the state synchronization set C_s is locally asymptotically stable for (5).

Proof: The proof uses some facts on reductive homogeneous spaces; we refer the reader to e.g. [4] for details. Let Δ the subgroup $\{(Q, \dots, Q) \mid Q \in SO(n)\}$ of $SO(n)^k$. Define the compact homogeneous space $M = SO(n)^k / \Delta$, with Δ acting on $SO(n)^k$ by right multiplication, and canonical projection $\pi \colon SO(n)^k \to M$. Since $SO(n)^k$ is compact, M is a reductive homogeneous space and we equip M with the Riemannian metric induced by the product metric on $SO(n)^k$.

Assume $\varepsilon = 1$. Since (5) and (6) equivariant under the action of $\Delta = \{(Q, \dots, Q) \mid Q \in SO(n)\}$ on $SO(n)^k$ by right multiplication, they respectively induce a time-varying vector field $F(x, t)$ and a time-invariant vector field $\overline{F}(x)$ on M. Note that for any Δ-equivariant, time-varying vector field $X(Q, t)$ on $SO(n)^k$ one has

$$T_Q \pi \int_{t_1}^{t_2} X(Q, t) dt = \int_{t_1}^{t_2} T_Q \pi(X(Q, t)) dt$$

where the integral is taken on the tangent spaces (with fixed Q). This means that vector field $\overline{F}$ is the time average of F. The image of state synchronization set C_s under the canonical projection π is a single point which we denote by $p \in M$. For both vector fields, thanks to their equivariance on $SO(n)^k$, point $p \in M$ has the same stability properties as set $C_s \subset SO(n)^k$. Thus p is exponentially stable under $\overline{F}$. Let $M_{ij}(t) = y_{ij}(t) y_{ij}(t)^T$ and characterize the difference between actual and averaged system by

$$\| \operatorname{skew}(Q_i^T \overline{M}_{ij} Q_j) - \operatorname{skew}(Q_i^T M_{ij}(t) Q_j) \|^2$$

A calculus argument shows that there is a positive continuous function $\phi \colon \mathbb{R}^{n \times n} \to \mathbb{R}$ such that, for $(Q_1, \dots, Q_k) \in SO(n)^k$,

$$\begin{aligned} &\| \operatorname{skew}(Q_i^T \overline{M}_{ij} Q_j) - \operatorname{skew}(Q_i^T M_{ij}(t) Q_j) \|^2 \\ &\quad \leq \phi(\overline{M}_{ij} - M_{ij}(t)) \ \operatorname{dist}_E((Q_1, \dots, Q_k), C_s)^2, \end{aligned}$$

where dist_E denotes the Euclidean distance in $\mathbb{R}^{k(n \times n)}$. Since $M_{ij}(t)$ belongs to a compact set for all $t \in \mathbb{R}$, there is a uniform upper bound $c_1 > 0$ for $\phi(\overline{M}_{ij} - M_{ij}(t))$, such that

$$\begin{aligned} &\| \operatorname{skew}(Q_i^T \overline{M}_{ij} Q_j) - \operatorname{skew}(Q_i^T M_{ij}(t) Q_j) \|^2 \\ &\quad \leq c_1 \operatorname{dist}_E((Q_1, \dots, Q_k), C_s)^2. \end{aligned}$$

Denoting the time-varying vector field (5) on $SO(n)^k$ by $F_{SO}(Q_1, \dots, Q_k, t)$ and the averaged one (6) by

$\overline{F}_{SO}(Q_1,\ldots,Q_k)$, we have

$$\left\|\int_{t_1}^{t_2}\overline{F}_{SO}(Q_1,\ldots,Q_k)-F_{SO}(Q_1,\ldots,Q_k,t)dt\right\|\leq$$
$$\sum_{i,j=1}^{k}a_{ij}\int_{t_1}^{t_2}\|\operatorname{skew}(Q_i^T\overline{M}_{ij}Q_j)-\operatorname{skew}(Q_i^TM_{ij}(t)Q_j)\|dt$$
$$\leq\sum_{i,j=1}^{k}a_{ij}(t_2-t_1)c_1^{1/2}\operatorname{dist}_E((Q_1,\ldots,Q_k),C_s)$$
$$=\#E\;c_1^{1/2}\;(t_2-t_1)\;\operatorname{dist}_E((Q_1,\ldots,Q_k),C_s).$$

Projecting from $SO(n)^k$ to M we get

$$\left\|\int_{t_1}^{t_2}\overline{F}(x)-F(x,t)dt\right\|\leq\#Ec_2(t_2-t_1)\;\operatorname{dist}(x,p)$$

where dist is now the Riemannian distance and $c_2>0$ a suitable constant. Using local charts around p we see now that the conditions of Theorem 3 in [2] hold for pair $(F,\overline{F})$ and hence the time-varying system $F(x,\delta t)$ is asymptotically stable for sufficiently large δ. A change of timescales yields asymptotic stability of C_s for the time-varying system (5) for sufficiently small ε. ■

One can give quantitative estimates for a sufficiently small ε based on the averaging theory [2]. However, these rather technical issues are beyond the scope of this paper.

V. Application: attitude synchronization from relative position measurements

To illustrate our theory, we consider the problem of synchronizing the *attitudes* of k rigid bodies which only measure relative *positions* in body frame. This setting is proposed in [17].

The attitude of each rigid body is given by $Q_i\in SO(3)$, the transformation from the body fixed frame into an arbitrary common inertial frame. In addition, the rigid bodies have positions $p_1(t),\ldots,p_k(t)\in\mathbb{R}^3$, which can be constant or time-varying. If agent i is connected to agent j, it observes, e.g. by an onboard camera, the direction of the relative position of j in its body fixed frame; this observation corresponds to $Q_i^Ty_{ij}$ where

$$y_{ij}=(p_j-p_i)/\|p_j-p_i\|. \qquad (7)$$

Then i sends this information to j, which itself sends $-Q_j^Ty_{ij}$ to i; the sign is easily corrected. Note that (7) introduces a linear dependence among the y_{ij} that is absent from our initial setting; thus results that are generic in the context of Sections III and IV are not necessarily generic here anymore.

Dynamics (1), $\frac{d}{dt}Q_i=Q_iu_i$, corresponds to assuming that control inputs are the angular velocities $\omega_1^i,\omega_2^i,\omega_3^i$ in body frame, with

$$u_i=\begin{pmatrix}0&-\omega_3^i&\omega_2^i\\\omega_3^i&0&-\omega_1^i\\-\omega_2^i&\omega_1^i&0\end{pmatrix}.$$

Feedback law (3) becomes

$$u_i=\sum_{j=1}^{k}\frac{a_{ij}}{\|p_j-p_i\|^2}\operatorname{skew}((p_j-p_i)^TQ_iQ_j^T(p_j-p_i)).$$

Thanks to the invariance properties of the feedback law, it is implementable in body frame, i.e. without requiring a common inertial frame.

Simulation results of our output feedback control law on $SO(3)$ with $k=6$ are represented on Figure 1. Interconnection among agents is taken to be all-to-all. Initial orientations $Q_i(0)$, $i=1\ldots k$, are independently randomly chosen in $SO(3)$. Each plot shows the time evolution of the maximal output error $\frac{1}{2}\|Q_i^Ty_{ij}-Q_j^Ty_{ij}\|^2$ (faint red curve) and of the maximal state error $\|Q_i-Q_j\|$ (thick blue curve) among all agent pairs. Initial conditions are randomly chosen in the whole state space (i.e. not restricted to a neighborhood of synchronization).

For Figure 1.**a**, the y_{ij} are defined by (7) with randomly independently chosen *fixed* p_i. Output synchronization is quickly reached, but orientations (i.e. states) only converge towards each other very slowly. Note that defining y_{ij} by (7) leads to a particular case where condition of Theorem 2 is never satisfied and locally exponential convergence is not guaranteed.

In contrast, if the y_{ij} are not restricted by (7), i.e. not relative positions, it appears that state synchronization is quickly reached for $k\geq 4$ fully connected agents; this is illustrated on Figure 1.**b** with $k=6$ agents. Numerical experiments indicate that the condition of Theorem 2 is generically satisfied for $k\geq 6$.

Finally, Figure 1.**c** again imposes (7) but with quasi-periodically varying positions $p_i=p_{i1}(1+\cos(t))+p_{i2}\cos(0.3t)+p_{i3}\cos(0.7t)$ for randomly chosen $p_{i1},p_{i2},p_{i3}\in\mathbb{R}^3$. State synchronization is recovered. Note that this observation is made with a reasonable frequency for the time-varying p_i.

VI. Conclusion

In this paper we considered the problem of distributed synchronization for agents whose states evolve on $SO(n)$ and which exchange only partial state information along communications links. We proposed a gradient algorithm based on a cost function on the output space of the system. For fixed output maps this algorithm locally converges to the set of output synchronization states. We discussed the relationship between output synchronization and state synchronization and gave a sufficient condition for output synchronization to coincide with state synchronization with exponential stability of the state synchronization set. For time-varying output maps, we used an averaging approach to prove that local convergence to the set of state synchronization states is obtained with less communication links and weaker conditions on the instantaneous output map. The algorithm is illustrated on an attitude synchronization problem where agents exchange only their relative positions observed in body frames.

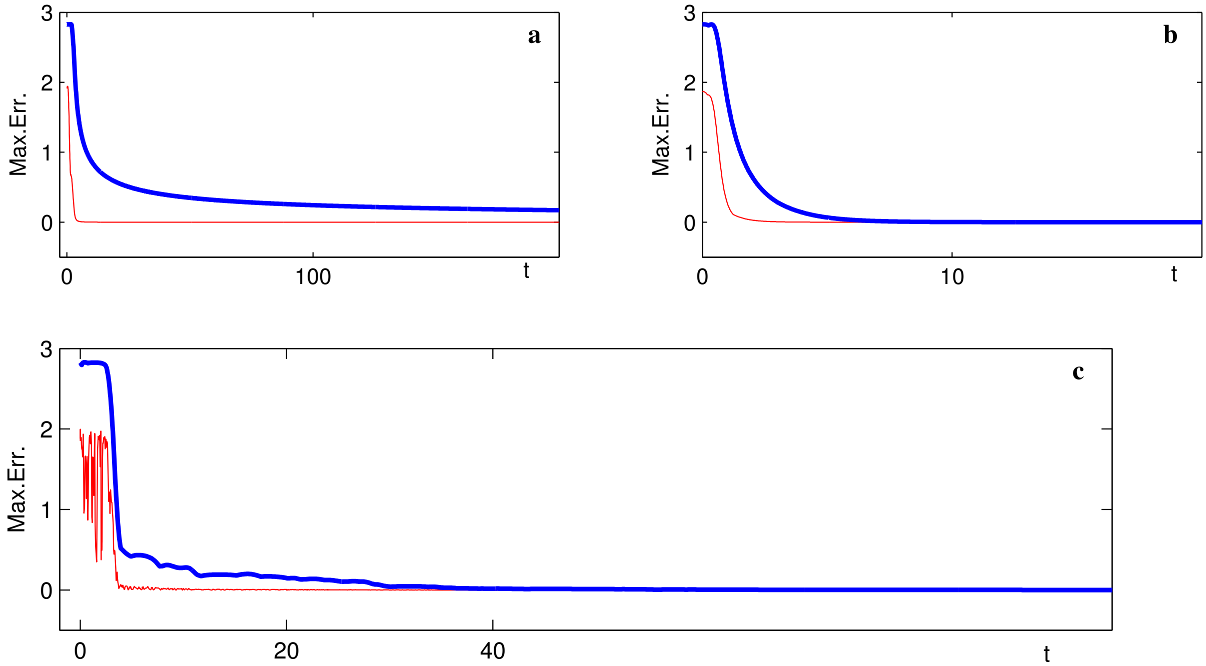

Fig. 1. Maximal output error (faint red) and state error (thick blue) among pairs of agents applying (3) on $SO(3)$ for $k = 6$. **a**: Fixed y_{ij} defined by (7). **b**: Fixed y_{ij} not restricted to (7). **c**: Time-varying y_{ij} defined by (7).

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REFERENCES

[1] P.-A. Absil and K. Kurdyka. On the stable equilibrium points of gradient systems. *Systems Control Lett.*, 55(7):573–577, July 2006.

[2] D. Aeyels and J. Peuteman. On exponential stability of nonlinear time-varying differential equations. *Automatica*, 35:1091–1100, 1999.

[3] A. Bondhus, K. Pettersen, and J. Gravdahl. Leader/follower synchronization of satellite attitude without angular velocity measurements. *Proc. 44th IEEE Conf. Decision and Control*, pages 7270–7277, 2005.

[4] J. Cheeger and D. G. Ebin. *Comparison theorems in Riemannian geometry*. North-Holland, Amsterdam, 1975.

[5] J. Cortés, S. Martínez, and F. Bullo. Coordinated deployment of mobile sensing networks with limited-range interactions. *Proc. 43rd IEEE Conf. Decision and Control*, pages 1944–1949, 2004.

[6] U. Helmke and J. Moore. *Optimization and dynamical systems*. Springer, 1994.

[7] J. Hendrickx, B. Anderson, J. Delvenne, and V. Blondel. Directed graphs for the analysis of rigidity and persistence in autonomous agents systems. *International journal of robust and nonlinear control*, 17(10):960–981, 2007.

[8] J. Hopfield. Neural networks and physical systems with emergent collective computational capabilities. *Proc. Nat. Acad. Sci.*, 79:2554–2558, 1982.

[9] A. Jadbabaie, J. Lin, and A. Morse. Coordination of groups of mobile autonomous agents using nearest neighbor rules. *IEEE Trans. Automatic Control*, 48(6):988–1001, 2003.

[10] E. Justh and P. Krishnaprasad. A simple control law for UAV formation flying. Technical report, TR 2002-38, ISR, University of Maryland, 2002.

[11] T. Krogstad and J. Gravdahl. Coordinated attitude control of satellites in formation. In *Group Coordination and Cooperative Control*, volume 336 of *Lect. N. Control and Information Sci.*, chapter 9, pages 153–170. Springer, 2006.

[12] J. Lawton and R. Beard. Synchronized multiple spacecraft rotations. *Automatica*, 38:1359–1364, 2002.

[13] N. Leonard, D. Paley, F. Lekien, R. Sepulchre, D. Frantantoni, and R. Davis. Collective motion, sensor networks and ocean sampling. *Proc. IEEE*, 95(1):48–74, 2007.

[14] C. McInnes. Distributed control for on-orbit assembly. *Adv. Astronautical Sci.*, 90:2079–2092, 1996.

[15] S. Nair. Stabilization and synchronization of networked mechanical systems. *PhD Thesis, Princeton University*, 2006.

[16] R. Olfati-Saber, J. Fax, and R. Murray. Consensus and cooperation in networked multi-agent systems. *Proc. IEEE*, 95(1):215–233, 2007.

[17] G. Piovan, I. Shames, B. Fidan, F. Bullo, and B. D. O. Anderson. On frame and orientation localization for relative sensing networks. In *Proceedings of the 47th IEEE Conference on Decision and Control*, pages 2326–2331, 2008.

[18] W. Ren. Distributed attitude consensus among multiple networked spacecraft. *Proc. American Control Conference*, 2006.

[19] C. Reynolds. Flocks, herds, and schools: a distributed behavioral model. *Comp. Graph. (ACM SIGGRAPH Conf. Proc.)*, 21:25–34, 1987.

[20] A. Sarlette and R. Sepulchre. Consensus optimization on manifolds. *SIAM J. Control and Optimization*, 48(1):56–76, 2009.

[21] A. Sarlette, R. Sepulchre, and N. Leonard. Autonomous rigid body attitude synchronization. *Automatica*, 45(2):572–577, 2009.

[22] S. Strogatz. *Sync: The emerging science of spontaneous order*. Hyperion, 2003.

[23] J. Tsitsiklis. Problems in decentralized decision making and computation. *PhD Thesis, MIT*, 1984.

[24] S. Tuna. Synchronizing linear systems via partial-state coupling. *Automatica*, 44(8):2179–2184, 2008.

[25] M. VanDyke and C. Hall. Decentralized coordinated attitude control of a formation of spacecraft. *J. Guidance, Control and Dynamics*, 29(5):1101–1109, 2006.

[26] T. Vicsek, A. Czirók, E. Ben-Jacob, I. Cohen, and O. Shochet. Novel type of phase transition in a system of self-driven particles. *Phys. Rev. Lett.*, 75(6):1226–1229, 1995.