Achieving string stability with nonlinear control inspired by a PDE

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***Abstract*— This paper deals with the problem of string stability of unidirectionally interconnected systems with double-integrator open loop dynamics (e.g. acceleration-controlled vehicles). We propose a nonlinear coupling law derived as an appropriate discrete space approximation of a partial differential equation (PDE), namely the Korteweg-de Vries equation. We argue that string instability, an unavoidable shortcoming of linear controllers, could thus be avoided as the stable propagation behavior of the Korteweg-de Vries equation carries over to the discretized system. Besides some formal arguments in this sense, we present simulation results showing that our nonlinear controller favorably compares to a (failing) linear control and saturation-based nonlinearity. The results use a notion of string stability whose nonlinear implications are somewhat adapted.**

I. Introduction

Consider the task of coordinating N identical vehicles moving on a line, where each of them is separated by a small distance from its front and rear neighbors. The open-loop model of each vehicle is a double integrator, in accordance with positions as output and forces $\simeq$ accelerations as input. This platooning problem, commonly called the *vehicle chain*, is relevant e.g. for automated highway systems, as efficient control allows smaller inter-vehicular distance which improves the capacity of a highway. Hence, during the recent years numerous works have considered the vehicle chain problem with different control strategies [1], [2], [3], [4], [5], [6]. Most of the standard methods to design local controllers for the interconnected systems can guarantee input-to-output stability, but would lead to string instability.

Definition 1: The vehicle chain is *string stable* if given any $\epsilon > 0$ there exists a $\sigma > 0$ such that $\|u\|_{L_2} < \sigma$ ensures $\|y_N - y_{N-1} - d\|_{L_2} < \epsilon$ independently of the number of vehicles N, where u, y_i and d are respectively the perturbation injected at the input of the first vehicle, the position of the i^{th} vehicle, and the desired distance between consecutive vehicles.

Since its definition in [5], [7], string (in)stability has attracted a lot of discussion. Among others, it is well known since [5], [7] that string stability *cannot be achieved* in a homogeneous string of interconnected systems if the following assumptions are true:

- the dynamical controller does not cancel any of the two poles at zero;
- each vehicle is connected only to a restricted set of ‘nearest neighbors’ which is independent of N;
- the coupling is unidirectional , i.e. each vehicle’s behavior (input information) can only depend on *vehicles in front of itself*;
- relative measurements, i.e. each vehicle i can only measure its relative position $y_i - y_j$ to other vehicles j, from which it can get a noisy estimate of relative velocity; however, it has no access to its absolute position y_i or velocity $\dot{y}_i$;
- the control law is linear and time-invariant.

When the ‘set of neighbors’ reduces to one directly preceding vehicle, a quick proof of this fact follows essentially a variation of the Bode integral for the transfer function from vehicle i to $i+1$; due to the relative measurements, the latter indeed takes the form of a sensitivity function. For bidirectional coupling, the connection to only one preceding and one following vehicle has also been shown to unavoidably imply string instability in conjunction with the other above assumptions [8].

To achieve string stability, one of these assumptions at least must be relaxed. We should keep the first one, because it is well known that pole cancellation leads to systems that are not robust. The second assumption is essential for a controller that is scalable to large networks. Several researchers have relaxed the unidirectional coupling constraint, yet from the result of [8] one would doubt if this can solve the problem. We here want to keep unidirectionality, for the sake of information flow from front to rear of the chain that is naturally present in moving vehicles. The fourth assumption, relative measurements, has been relaxed in a “time headway policy” formulation, see [11], [12], [6] and other recent papers. Time headway is measured as the distance from the tip of one vehicle to the tip of the one behind it, expressed as the time it will take for the trailing vehicle to cover this distance. Computing time headway in a controller requires to know the absolute velocity (although still not the absolute position) of the trailing vehicle. The authors prove that, while keeping all the other assumptions above, using a sufficiently large time headway allows to bound independently of N the norm of the transfer function from error signal on the first vehicle to induced disturbance on vehicle N. The use of absolute velocity in the controller requires an external static reference system common to all the vehicles. There

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are some situations where this is not so obvious (GPS outage, underwater, aerial, space flight) and disposing of this additional measurement through more efficient control would make sense.

Hence in the present paper, we relax the last assumption. The robustness of *nonlinear* controllers relying on a saturation term has been considered in [10] both for the bidirectional and unidirectional cases where the vehicles are interconnected with their nearest neighbors. Monte Carlo simulations show that the ratio from the input injected at the first vehicle to the motion of the last vehicle grows through the vehicle chain much more slowly than in the linear case; however a string stability analysis or related design principles are not addressed explicitly in [10] and in fact we see (see our comparison below) that the system is not "string stable", even according to our revised definition for nonlinear systems. In contrast, we derive a nonlinear control law from the Korteweg-de Vries equation (KdV) which is a nonlinear partial differential equation (PDE) of third order and provides an important example of a dispersive nonlinear wave process. This equation was first derived as a result of studies of shallow water waves [13]. Authors have characterized so-called *soliton solutions* of the KdV equation [15], [16], i.e. perturbations which propagate along the spatial direction without deformation, in particular with amplitude stable to small disturbances [16]. This property inspired us to use the KdV equation as a basis for a hopefully string stable system. KdV equation is also employed to model traffic jams in another context, where the absolute velocity of each vehicle is added to the controller [14]. We employ a nontrivial appropriate finite difference method looking only at vehicles in front to spatially discretize the KdV equation towards a nonlinear control law for a discrete set of vehicles with unidirectional coupling.

The paper is organized as follows. Section II briefly introduces a slightly relaxed definition for string stability of nonlinear control systems. In Section III we present the Korteweg-de Vries PDE and the vehicle chain model to be controlled. In Section IV we explain how a nonlinear controller for the vehicle chain is derived from the Korteweg-de Vries PDE. In Section V we analyze the resulting vehicle chain behavior with approximation arguments, and in Section VI we provide further evidence for string stability through simulations.

II. Re-definition of String Stability

The notion of string stability proposed in Definition 1 is not best suited for nonlinear systems. Indeed, requiring string stability for ϵ infinitesimal would confine the system to a very small neighborhood, where it should be equivalent to its linearization. Thus at that scale the impossibility result for linear systems applies, such that nonlinear effects can have no benefit towards string stability with infinitesimal ϵ. However for all practical purposes, being able to guarantee confinement to an a priori selected finite ϵ would appear as a reasonable robustness objective. We hence propose a new definition of string stability where the controller tuning is allowed to depend on ϵ (but still not on N!). For simplicity of notation in all the following, we also introduce a constant change of variables such that the desired reference is $y_k - y_{k-1} = 0$ for all k, e.g. in the vehicle chain we make the change of notation $y_k \to y_k - k \cdot d$.

Definition 2: A system parameterized by control parameters c is *string stable* if given any $\epsilon > 0$ there exists a $\sigma > 0$ and a controller tuning c, independent of the number of vehicles N, such that $\|u\|_{L_2} < \sigma$ ensures $\|y_N - y_{N-1}\|_{L_2} < \epsilon$.

For the linear case this is equivalent to Definition 1, since if $\|u\| < \sigma$ implies $\|y_N - y_{N-1}\| < \bar{\epsilon}$ for some particular c then $\|u\| < (\epsilon/\bar{\epsilon})\sigma$ implies $\|y_N - y_{N-1}\| < \epsilon$ for any $\epsilon < 0$ and the same c. In the nonlinear case however, $\|y_N - y_{N-1}\|$ might converge to some limit value that depends on c, but not on N nor essentially on the input disturbance σ — think of e.g. a stable limit cycle. If adapting c can scale down the amplitude of that limit arbitrarily, then we would accept the situation as string stable.

III. PDE and Vehicle Chain models

We formulate a PDE with only one spatial variable η because it is supposed in this paper that each vehicle only has interconnections with other vehicles on the same line e.g. on the highway. So the solution of the PDE will be a function u depending on the variables η and τ, i.e. $v(\eta, \tau)$ describes the corresponding value of the wave at place η and at time τ.

Using the typical short notations

$$v_\tau(\eta,\tau) = \frac{\partial v(\eta,\tau)}{\partial \tau}; v_\eta(\eta,\tau) = \frac{\partial v(\eta,\tau)}{\partial \eta}$$

$$v_{\eta\eta}(\eta,\tau) = \frac{\partial^2 v(\eta,\tau)}{\partial^2 \eta}; v_{\eta\eta\eta}(\eta,\tau) = \frac{\partial^3 v(\eta,\tau)}{\partial^3 \eta}$$

the Korteweg-de Vries (KdV) equation writes:

$$v_\tau(\eta,\tau) + \beta v(\eta,\tau) v_\eta(\eta,\tau) + \gamma v_{\eta\eta\eta}(\eta,\tau) = 0\,. \quad (1)$$

This PDE is nonlinear because of the product in the second summand and is of third order through the third summand. The parameters β and γ are just scaling factors. The KdV equation (1) has propagating soliton solutions of the form

$$v(z) = \pm\frac{3C}{\beta} sech^2(3z\frac{\sqrt{C\gamma}}{\beta} + \eta_0) \quad (2)$$

where $z = \eta - C\tau$, with propagation velocity C and η_0 depending on the initial condition and boundary behavior at $\eta = 0$. Note that more "peaked" solitons also propagate faster.

We target this "string stable" pure transport PDE behavior as a closed-loop model for the vehicle chain — at least for the solitons with values of C smaller than some bound, which are approximable by a space discretization. Considering a chain of N vehicles, the spatial independent variable η is replaced by the discrete vehicle index $i = 1, 2, ..., N$, while the output $y_i(t)$ at index i denotes the i^{th} vehicle's position at time t, replacing $y(\eta,\tau)$. We model each vehicle as a double integrator:

$$\ddot{y}_i = u_i \tag{3}$$

where, in a decentralized control architecture with unidirectional coupling, the control input u_i (force) applied to the i^{th} vehicle should depend only on its relative position and relative velocity with respect to a few preceding vehicles. The transition from PDE to vehicle chain requires an appropriate discretization.

IV. Nonlinear control design from the Korteweg-de Vries equation

In this section we provide a nonlinear control design procedure by making the link from KdV equation to the chain of vehicles towards satisfying string stability. Furthermore, we prove how to adapt the coefficients of the resulting nonlinear controller in order to attain an arbitrarily small $\epsilon > 0$ in Definition 2, provided the strategy works for some given $\bar{\epsilon} > 0$.

A. Nonlinear Control Design Procedure

The following procedure is inspired by arguments in [18, p.49-p.50]. To link the KdV equation to the vehicle chain, we must get a second-order time derivative. To this end we make a first substitution $v = y_\eta$, for which the KdV equation (1) implies:

$$y_{\eta\tau}(\eta,\tau) + \beta y_\eta(\eta,\tau) y_{\eta\eta}(\eta,\tau) + \gamma y_{\eta\eta\eta\eta}(\eta,\tau) = 0\,. \tag{4}$$

We then define a change of variables $\eta = x + h\sqrt{\omega}t$ and $\tau = \frac{h^3}{12\sqrt{\omega}}t$, where the roles of h and ω will be specified later. Then by the chain rule $\frac{\partial^k}{\partial x^k} = \frac{\partial^k}{\partial \eta^k}$, $\frac{\partial}{\partial t} = h\sqrt{\omega}\frac{\partial}{\partial \eta} + \frac{h^3}{12\sqrt{\omega}}\frac{\partial}{\partial \tau}$, and $\frac{\partial^2}{\partial t^2} = h^2\omega\frac{\partial^2}{\partial \eta^2} + \frac{h^4}{12}\frac{\partial^2}{\partial \eta\partial\tau} + (\frac{h^6}{24^2\omega})\frac{\partial^2}{\partial \tau^2}$. To facilitate later expressions we further define $\beta' = \frac{h\beta}{24}$. In these new coordinates, the equation

$$y_{tt} = h^2\omega y_{xx} - 2h^3\beta' y_x y_{xx} - \frac{h^4}{12}\gamma y_{xxxx}\,, \tag{5}$$

equals (4) up to a term proportional to $hy_{\tau\tau}$ which we will intentionally keep very small (see next section).

We discretize this equation in space with an appropriate finite difference method looking only at vehicles in front. The parameter h is taken as discretization step to map the consecutive vehicles to points in x-space:

$$\begin{aligned} y_x &\simeq \frac{y_i - y_{i-1}}{h} \\ y_{xx} &\simeq \frac{(y_i - y_{i-1}) - (y_{i-1} - y_{i-2})}{h^2} \\ &\quad -\frac{h^2}{12}\Big(\frac{3(y_{i-2} - y_{i-3}) - (y_{i-3} - y_{i-4})}{h^4}\Big) \\ y_{xxxx} &\simeq \frac{(y_i - y_{i-1}) - 3(y_{i-1} - y_{i-2})}{h^4} \\ &\quad +\frac{3(y_{i-2} - y_{i-3}) - (y_{i-3} - y_{i-4})}{h^4}\,. \end{aligned} \tag{6}$$

Substituting this into (5) we get:

$$\begin{aligned} \ddot{y}_i \;=\; & \frac{(\gamma+\omega)}{12}\left[-(y_i - y_{i-1}) + 3(y_{i-1} - y_{i-2})\right. \\ & \left. - 3(y_{i-2} - y_{i-3}) + (y_{i-3} - y_{i-4})\right] \\ & +\omega\left[(y_i - y_{i-1}) - (y_{i-1} - y_{i-2})\right] \\ & -2\beta'\left[(y_i - y_{i-1})^2\right. \\ & \left. - (y_i - y_{i-1})(y_{i-1} - y_{i-2})\right] \end{aligned} \tag{7}$$

for $i = 5, 6, ..., N$. The resulting control law clearly satisfies the assumptions of local control, relative measurements, no poles cancellation, and unidirectional coupling for a second-order integrator system. The KdV equation features a large invariant space, so to damp relative modes we may add a term proportional to relative velocity:

$$\begin{aligned} \ddot{y}_i \;=\; & \frac{(\gamma+\omega)}{12}\left[-(y_i - y_{i-1}) + 3(y_{i-1} - y_{i-2})\right. \\ & \left. - 3(y_{i-2} - y_{i-3}) + (y_{i-3} - y_{i-4})\right] \\ & +\omega\left[(y_i - y_{i-1}) - (y_{i-1} - y_{i-2})\right] \\ & -2\beta'\left[(y_i - y_{i-1})^2\right. \\ & \left. - (y_i - y_{i-1})(y_{i-1} - y_{i-2})\right] \\ & -b\left(\dot{y}_i - \dot{y}_{i-1}\right). \end{aligned} \tag{8}$$

Since each vehicle following (8) requires information from up to 4 vehicles in front, other evolution laws must be given for the 4 first vehicles. These other laws have no influence on string stability, as long as they lead to L_2-bounded signals y_2, y_3, y_4 for any L_2-bounded signal y_1.

B. Effective controller tuning

The following fact is a direct consequence of a rescaling of variables in (8).

Proposition 1: Consider the nonlinear controller (8) with $\beta' = \beta_1$ and the four leading vehicles undergoing motions $\bar{y}_1(t), ..., \bar{y}_4(t)$, and denote by $\bar{y}_N(t)$ the corresponding solution for the motion of vehicle N starting from zero-error initial conditions. Then the nonlinear controller (8) with $\beta' = \alpha\beta_1$ and leading vehicles undergoing motions $\bar{y}_1(t)/\alpha, ..., \bar{y}_4(t)/\alpha$ features the solution $\bar{y}_N(t)/\alpha$ for the motion of vehicle N starting from zero-error initial conditions.

This result implies that if the controller in the first case satisfies $\|\bar{y}_N\| < \epsilon'$ with some possibly large $\epsilon' > 0$, for all N and all leading vehicle motions with $\|\bar{y}_i\| < \sigma$, $i = 1, ..., 4$; then by multiplying by α the gain associated to the nonlinear term in the controller we can ensure $\|\bar{y}_N\| < \epsilon'/\alpha$ when $\|\bar{y}_i\| < \sigma/\alpha$, $i = 1, ..., 4$, for any $\alpha > 1$. Thus for any *imposed* $\epsilon > 0$, we can find a value of α and a bound on allowed leading vehicles' disturbances such that the system satisfies $\|\bar{y}_N\| < \epsilon'/\alpha = \epsilon$ for all N. In other words, we have shown that we can make the system string stable according to Definition 2, provided for some controller gain it is "string bounded". Although a full proof of the latter condition for (8) is currently elusive, we provide several supporting arguments in the remainder of the paper.

V. Stability of the resulting Vehicle Chain System

In this section we provide a partial analysis of the discrete space system described in (7) or (8). More precisely, we combine results on the KdV equation, which we approximate for slowly varying signals, and a linearized error equation investigation for high frequency signals. Strictly speaking this is not a full analysis of string stability — especially since (i) the superposition principle does not hold for the nonlinear equation (8) and (ii) the definition of string stability requires to check *all* L_2-bounded signals. However, the following arguments, in conjunction with the simulation results of Section (VI), are good indications that the KdV approximation approach improves on all linear controllers.

A. Slowly varying perturbations

The objective of this section is to argue that (7) is a good approximation of the KdV equation for slowly varying boundary condition i.e. motions of the leading vehicles. Hence, the string stability induced by the stable transport behavior of the KdV dynamics should hold, at least for those signals.

To investigate the approximation, we go back from (7) to the PDE but keeping all error estimations in the Taylor developments. Using Taylor's development in (7) directly yields

$$\begin{aligned} y_{tt} &= \omega(h^2 y_{xx} + \frac{h^4}{12} y_{xxxx} + \frac{h^6}{360} y_{xxxxxx} + O(h^8)) \qquad (9)\\ &\quad -(\gamma+\omega)(\frac{h^4}{12} y_{xxxx} + \frac{h^6}{360} y_{xxxxxx} + O(h^8))\\ &\quad -\beta'(2h^3 y_x y_{xx} + \frac{h^5}{6} y_x y_{xxxx} + \frac{h^5}{3} y_{xx} y_{xxx} + O(h^7)) \end{aligned}$$

which is relevant *if we select a small value for h, compared to the characteristic length of the solutions $y(x,t)$ of the PDE corresponding to a particular boundary condition.* It is in this sense that the leading vehicles must induce slow perturbations.

Working back the change of variables $(\eta,\tau) \leftrightarrow (x,t)$, equation (9) assumes the form

$$\begin{aligned} 0 \;=\; & y_{\tau\eta} + \frac{24\beta'}{h} y_\eta y_{\eta\eta} + \gamma y_{\eta\eta\eta\eta} \qquad (10)\\ & +2h\beta' y_\eta y_{\eta\eta\eta\eta} + 4h\beta' y_{\eta\eta} y_{\eta\eta\eta}\\ & +\frac{h^2}{30}\gamma y_{\eta\eta\eta\eta\eta\eta} + \frac{h^2}{12\omega} y_{\tau\tau} + O(h^3)\,. \end{aligned}$$

The first line of (10) represents the KdV equation with $v = y_\eta$. Hence we want to show that under some conditions, the other terms can be seen as small perturbations.

We will therefore assume that the system follows a soliton solution of the KdV equation. Since the soliton solution is a function of $z = \eta - C\tau$, the order of magnitude of $y_{\tau\tau}$ is C times the order of magnitude of $y_{\tau\eta}$. For the corresponding terms in (10), we have $\frac{h^2}{12\omega} y_{\tau\tau}$ of order Ch^2/ω times $y_{\tau\eta}$. Hence the term in $y_{\tau\tau}$ can be viewed as a small perturbation with respect to the KdV term in $y_{\tau\eta}$ provided Ch^2 is small. Similarly, plugging in the form (2), we can establish that the order of magnitude of (X_1) is (X_2) times the magnitude of (X_3), with respectively

$$\begin{aligned} ((\mathrm{X}_1),(\mathrm{X}_2)(\mathrm{X}_3)) \;\in\; \{ \;& (y_\eta y_{\eta\eta\eta\eta},\, \frac{Ch}{\beta'},\, y_{\eta\eta\eta\eta})\\ & (y_{\eta\eta} y_{\eta\eta\eta},\, \frac{Ch^2}{\beta'^2},\, y_\eta y_{\eta\eta})\\ & (y_{\eta\eta\eta\eta\eta\eta},\, \frac{Ch^3}{\beta'^3},\, y_\eta y_{\eta\eta})\;\}\,. \end{aligned}$$

Hence if we take Ch and h sufficiently small, in equation (10) the associated perturbative terms (X_1) are all strongly dominated by a (X_3) term from the target KdV equation.

The theory of singular perturbations on PDEs would have to be used to rigorously justify that we can neglect such dominated terms. However, it is current practice in PDE discretization that such terms can be safely neglected for stable schemes. Involving a higher number of neighboring vehicles in the control law would allow better approximations in this sense; in the PDE discretization this would be called a scheme of higher order.

The fact that the motion of the leading vehicle(s) must belong to L_2 readily excludes singular soliton solutions of the KdV equation [16] and therefore a typical solution takes the form (2). Moreover, it has been proved [15], [16] that (2) describes a stable solution of the KdV equation, as effects of dispersion and nonlinearity cancel each other. Linearization shows that the discretized scheme (7) is also a stable system for input y_1 (although maybe not string stable, i.e. the transfer function to y_N is bounded but not independently of N). Under these conditions, we argue that it is legitimate to say:

Proposition 2: For a given β', γ, ω in the vehicle chain control law (7), there exist sufficiently small $\bar{h} > 0$ and $\bar{C} > 0$ such that: If the motion of the leading vehicles is compatible with a (succession of) soliton solution(s) (2) with $|C| < \bar{C}$ of the KdV equation (1) with $\beta > 24\beta'/\bar{h}$, then we can expect the vehicle chain (7) to approximate a solution implied by the KdV soliton behavior.

It is worth detailing what the change of variables actually implies on the vehicles for a soliton solution of (1). The actual requirements come down to having a small C/β, small $1/\beta$ and small h. First, this implies a small amplitude of the soliton, which means that the controller, with fixed values of β', γ and ω, must be written in coordinates for y in which perturbations are expected to be small (e.g. kilometers). Second, it implies a slow variation of the soliton profile as a function of z. The change of variables yields

$$z = x + (h\sqrt{\omega} - Ch^3/(12\sqrt{\omega}))t\,,$$

where individual vehicle indices $i = 0, 1, 2, ...$ correspond to $x = 0, h, 2h, ...$ Hence as a function of space, we retrieve the intuitive condition of good discretization, namely that a characteristic length of the soliton should cover a large number of vehicles. The velocity of propagation between vehicles is $|(\sqrt{\omega} - Ch^2/(12\sqrt{\omega}))|$, hence dominated by $\sqrt{\omega}$. As a function of time, the variation of the state of an individual vehicle is thus *slow*, exclusively through the small variation of the soliton solution between neighboring vehicles.

A surprising effect of the change of variables is that those slow solitons propagate from back to front of the chain, i.e. a constant z means that x *decreases* as t increases. Hence a starting "bump" of the leading vehicle would not even propagate through the whole chain, but rather be evacuated readily through the first vehicles. This surprising behavior is confirmed in simulations of (7), see Fig. 1. This can be understood as a very special reaction of the following vehicles, which somehow "predict" a future soliton motion of the leaders. Correctly predicting solitons, and *only* solitons, on the basis of a small interval of the leaders' trajectory, might seem counterintuitive and fragile but it is certainly not logically impossible. Indeed nothing prevents a short interval of the leaders' trajectory from being uniquely mapped to timings and amplitudes of solitons.

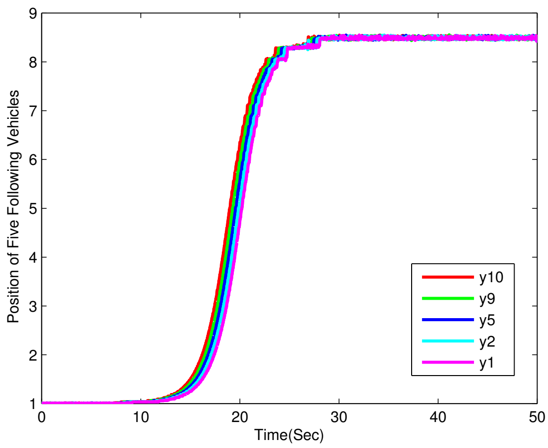

Fig. 1. Motion of ten following vehicles under control law (8) when the leading vehicles follow an integrated soliton solution, which takes the form $atanh(\cdot)$. The soliton appears to move backward as the nonlinear reaction of followers transforms ever tinier soliton-compatible fluctuations of their leaders into solitons that switch ever earlier.

B. Fast varying perturbations

Quick motions of the leading vehicles would not be compatible with the well-approximated soliton solutions of the KdV equation. This is in fact favorable since at high frequencies perturbations shall actually be damped, as we show next. The physical situation — e.g. considering the related power — justifies to expect that high-frequency variations of the leading vehicles would have small amplitude; hence the present subsection considers a linearized model.

The linearized dynamics of $e_i = y_i - y_{i-1}$ under the nonlinear controller (8) write in the Laplace domain:

$$\delta e_i(s) \;=\; \sum_{k=1}^{4} T_k(s)\, \delta e_{i-k}(s) \tag{11}$$

$$= \sum_{k=1}^{4} \frac{P(s)Q_k(s)}{1+P(s)(bs+\frac{(\gamma-11\omega)}{12}+2\beta'(2a_i-a_{i\text{-}1}))}\delta e_{i-k}(s)$$

for $i = 5, 6, \ldots$. Here $P(s) = 1/s^2$ and $Q_k(s)$ are a first-order polynomial for $k = 1$ and some constants for $k > 1$, as we linearize with $\delta e_i = e_i - \bar{e}_i$ around $\bar{e}_i$, approximating $a_i := \bar{e}_i - \bar{e}_{i-1}$ as a static situation. The determining factor

$$\frac{T_k(s)}{Q_k(s)} = \frac{1}{s^2+bs+(\frac{(\gamma-11\omega)}{12}+2\beta'(2a_i-a_{i\text{-}1}))}$$

characterizes a stable transfer function, provided $b \geq 0$ and an additional condition $\frac{(\gamma-11\omega)}{24\beta'} > a_{i-1} - 2a_i$ which relates our controller tuning and the expected state around which we linearize the vehicle chain. The conditions for Proposition 2 favor a *lower* value of β'. Hence given a typical bound on $|a_{i-1} - 2a_i|$, it is plausible to satisfy the conditions for stability of the linearized system by choosing β' not too large and $\gamma - 11\omega > 0$ large enough.

The norm of the T_k can be larger than 1 at low frequencies (e.g. see $s = 0$ and $a_i = 0$), implying string instability *if this linear part was the full control law*. At high frequencies, the T_k all converge to zero (as $1/s$ for T_1 and as $1/s^2$ for T_2, T_3, T_4 for $s \to \infty$). This is the typical low-pass-filter situation of linear controllers: at high frequencies there is no string stability problem, but at low frequencies it is unavoidable. However, if a nonlinear system is string unstable, then the linear approximation becomes invalid as a perturbation grows along the chain, such that at those (low) frequencies the nonlinear dynamics must be considered. For the latter, Section V-A argues that a string stable KdV-like behavior is obtained. We can thus conclude:

Proposition 3: For small high-frequency perturbations of the leading vehicles, the control law (7) is string stable, from its linearized form, provided $b \geq 0$, $\gamma' > 12\omega$, and β' is chosen small enough with respect to a bound on $|3\bar{e}_{i-1} - \bar{e}_{i-2} - 2\bar{e}_i|$ in the state around which we linearize. For low-frequency soliton-compatible perturbations, Proposition 2 applies and guarantees string stability.

The condition involving β' might at first case appear incompatible with Proposition 1. This is however not true: we are here analyzing the situation for a given β'; and once this is settled, Proposition 1 *guarantees exactly* the same but rescaled behavior when β' is modified.

A full proof of string stability would require to show that those or similar arguments can cover all cases of leader behaviors, or to apply another adapted method for nonlinear systems. This is ongoing work. The following simulations motivate research towards such a proof.

VI. Simulation

The approximations made in the previous sections call for a thorough checking of actual results. We will compare the performance of our nonlinear unidirectional control law (8) looking only at four vehicles in front, to the saturation-based controller of [10] and to a linear unidirectional control architecture which typically uses the relative position of one vehicle in front, e.g.

$$\ddot{y}_i = -k(y_i - y_{i-1}) - b\,(\dot{y}_i - \dot{y}_{i-1})\,, \tag{12}$$

where k, b are positive constants. We recall (see Section I) that *any* such linear controller looking at an *arbitrary* fixed number of preceding vehicles would provably be string unstable.

To formally show that the nonlinear controller is string stable, we would have to simulate its behavior for every L_2-bounded input signal. This is clearly not possible, but it seems reasonable to expect a relevant result if we select the input perturbations to be random — after all, if the system is "robustly string unstable", a component that is amplified along the chain should emerge. For the linear system, the situation is clearer: a random input shall contain a frequency that is amplified along the chain, and which remains by linearity whatever the rest of the signal. Anyways, in the simulation we cannot verify the definition of string stability but only check how a necessarily finite signal seems to get amplified along a necessarily finite chain. If any, that result should be relevant for practical purposes.

Hence in our simulations, random motions are assigned to the four leading vehicles denoted el_1 to el_4 as shown on Fig.2. The resulting motion errors of the following vehicles denoted by $e_1, e_2, e_{10}, ..e_{40}$ are shown on Figures 3 and 4 respectively for our nonlinear controller and for the linear PD controller. The parameters are chosen to be $\gamma = 200$, $\beta' = 80$, $\omega = 10$, $b = 1$; and $k = 50$.

On Fig.3, the signals y_k obtained with our nonlinear controller appear to stay bounded, even for smooth perturbations injected at the leading vehicles which clearly make the linear controller fail. Indeed as shown on Fig.4, the related signals y_k obtained with the linear controller clearly have an amplitude that increases with k; this is in accordance with the impossibility to avoid string stability with a linear controller. The low pass filter-type behavior of the controlled vehicle chains is also visible on the figures.

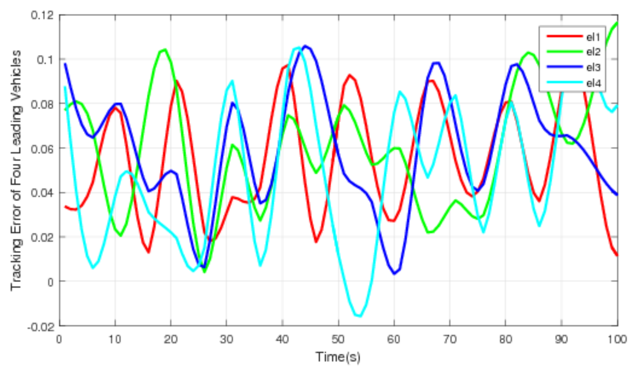

Fig. 2. Random perturbation signals assigned to the four leading vehicles.

To quantify the effect of random disturbances, we use like

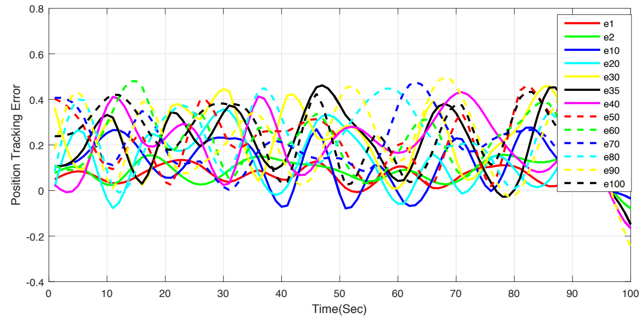

Fig. 3. Motion of following vehicles under our nonlinear unidirectional control law (8). The amplitude of the signal (visually) *stays bounded* when the index of the vehicle increases.

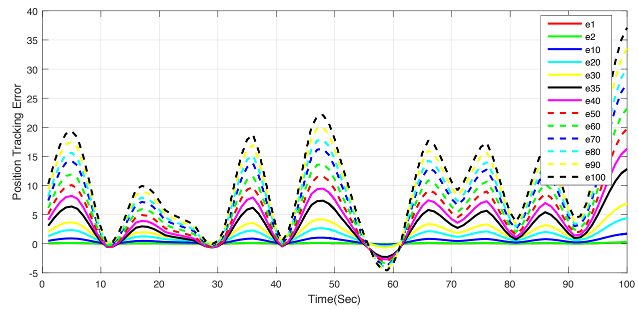

Fig. 4. Motion of following vehicles under the linear unidirectional control law (12). The amplitude of the signal (visually) *increases* when the index of the vehicle increases. It has been proven [5], [7] that this behavior is unavoidable for any linear controller.

in [10] the expected first-to-last ratio for random inputs

$$R_{FTL} := \frac{\sqrt{E(e_N^2(T))}}{\delta_0} \tag{13}$$

where $E(.)$ denotes the expectation over leading vehicle inputs, T is a sufficiently large time such that transients die out and δ_0 is the L_2 norm of the random input. In the linear case, R_{FTL} is exactly the H_2 norm of the transfer function from the random motion of first vehicle to the position tracking error of last vehicle. We have computed the estimate R_{FTL} for $T = 4000(s)$ through Monte Carlo simulations for both linear and nonlinear controllers. Figure 5 shows R_{FTL} versus number of vehicles N for a fixed δ_0 for the first four leading vehicles, for the parameter values $\gamma = 200$, $\beta' = 80$, $\omega = 10$, $b = 1$; and $k = 50$. For our nonlinear controller (8) the ratio appears to converge to a constant value, hinting at string stability, while in contrast the ratio for the linear controller (12) and for the saturation-based nonlinear controller of [10] keeps growing with N. We have furthermore checked (figure omitted due to space constraints) that taking e.g. $\beta' = 400$ and dividing the disturbance inputs by a factor five, the limit reached by (8) on Fig.5 is as well divided by five.

VII. Conclusion

We have proposed a nonlinear controller for a chain of vehicles with double-integrator dynamics. This controller is

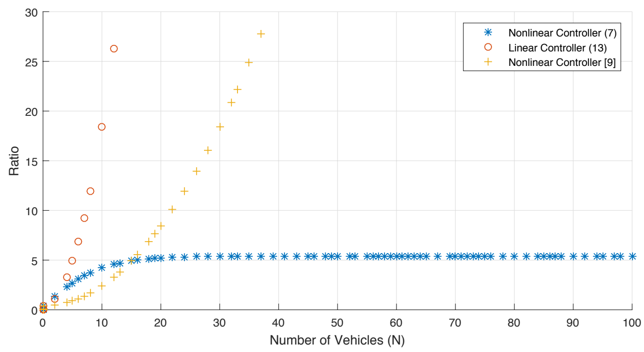

Fig. 5. First-to-last ratio R_{FTL}. Comparison of nonlinear and linear controllers of a network of 100 agents with random inputs of typical amplitude $\delta_0 = 0.5$.

inspired by the Korteweg-de Vries partial differential equation, to which we apply an appropriate coordinate change and a discretization which finally requires each vehicle only to have relative measurements with respect to the four closest front vehicles. We argue how this controller could avoid the string instability featured by *any* linear controller under the same circumstances. Although a full proof of string stability is currently lacking, simulation results confirm a promising behavior. Compared to the impossibility result in the literature, we hence seem to achieve string stability, by (i) using relative positions, (ii) looking only at a finite number of neighbors, (iii) looking only at vehicles in front, (iv) avoiding pole cancellation, but (v) replacing the linear by a nonlinear controller and slightly adapting the definition to allow this fact to become a game-changer. It is to inform the design of this nonlinear controller that we need to refer to the Korteweg-de Vries equation.

Preliminary investigations show that similar results can be obtained by discretizing other nonlinear partial differential equations. In the PDE-analogy context, of course control from boundary conditions is also an interesting research direction, see e.g. [17], although it is not our focus. The general thinking behind our paper is rather to show how partial differential equations can serve as inspiration for designing the *nonlinear coupling* in networked controllers in order to obtain particular behaviors (known from the typical behavior of the PDE). Besides for its stabilizing properties, this could be used to generate feedforward propagating signals — e.g. solitons with the Korteweg-de Vries equation, think of a vehicle chain that has to circumnavigate an obstacle on its path; or to get specific nonlinear responses to distributed stimuli — e.g. mimicking interesting phenomena taken from meta-materials. For completeness we must note that design of distributed systems' coupling from *linear* PDEs has already been proposed in several papers, amongst which [9] provides interesting conclusions towards vehicle chains (although not string stability nor the just mentioned possible features since the resulting controller is linear).

REFERENCES

[1] K. C. Chu, "Decentralized control of high-speed vehicular strings," *Transportation Science*, vol. 8, no. 4, pp. 361-384, 1974.

[2] S. Klinge, "Stability issues in distributed systems of vehicle platoons," Master's thesis, Otte-von-Guericke-University Magdeburg, *http:/www.hamilton.ie/publications.htm*, 2008.

[3] S. Sheikholeslam and C. Desoer, "Longitudinal control of a platoon of vehicles," *Proc. American Control Conf.*, pp. 291-297, 1990.

[4] W. Levine and M. Athans, "On the optimal error regulation of a string of moving vehicles," *IEEE Trans. Automatic Control*, vol. 11, no. 3, pp. 355-361, 1966.

[5] D. Swaroop and J. Hedrick, "String stability of interconnected systems," *IEEE Trans. Automatic Control*, vol. 41, no. 3, pp. 349-357, 1996.

[6] J. Ploeg, D.P. Shukla, N. van de Wouw and H. Nijmeijer, "Controller synthesis for string stability of vehicle platoons," *IEEE Trans.Intell.Transp. Systems*, vol. 15, no. 2, pp.854-865, 2014.

[7] D. Swaroop, "String stability of interconnected systems: An application to platooning in automated highway systems," *PhD thesis, University of California, Berkeley*, 1994.

[8] P. Barooah and J. P. Hespanha, "Error amplification and disturbance propagation in vehicle strings with decentralized linear control," *Proc. IEEE Conf. on Decision and Control*, pp. 4964-4969, 2005.

[9] P. Barooah, P. G. Mehta and J. P. Hespanha, "Mistuning-based control design to improve closed-loop stability of vehicular platoons," *IEEE Trans. Automatic Control*, vol. 54, no. 9, pp. 2100-2113, 2009.

[10] H. Hao and P. Barooah "Stability and robustness of large platoons of vehicles with double-integrator models and nearest neighbor interaction " *International Journal of Robust and Nonlinear Control*, vol. 23, pp. 2097-2122, 2013.

[11] C. Chien and P. Ioannou, "Automatic Vehicle following," *Proc. American Control Conf.*, pp. 1748-1752, 1992.

[12] S. Klinge, and R. H. Middleton, "Time headway requirements for string stability of homogenous linear unidirectionally connected systems," *Proc. IEEE Conf. on Decision and Control*, pp. 1992-1997, 2009.

[13] D. J. Korteweg and G. De Vries, "On the change of form of long waves advancing in a rectangular channel, and on a new type of long stationary waves," *Philosophical Magazine*, vol. 5, no. 39, pp. 422-443, 1895.

[14] J. Monteil, R. Billot, J. Sau, and N. E. El Faouzi "Linear and Weakly Nonlinear Stability Analyses of Cooperative Car-Following "*IEEE Trans. Intelligent Transportation Systems*, vol. 15, no. 5, pp. 2001-2013, 2014.

[15] H. J. S. Dorren and R. Snieder, "A stability analysis for the Korteweg-de Vries equation," *http://arxiv.org/abs/solv-int/9605005*, 1996.

[16] H. J. S. Dorren, "Singular and unstable solutions of the Korteweg-de Vries hierarchy," *Journal of Mathematical Physics*, vol. 37, 1996.

[17] T. Meurer and M. Krstic, "Finite-time multi-agent deployment: A nonlinear PDE motion planning approach," *Automatica*, vol. 47, no. 11, pp. 2534-2542, 2011.

[18] R.S. Palais, "An introduction to wave equations and solitons," *The Morningside Center of Mathematics, Chinese Academy of Sciences, Beijing*, 2000.