Stabilizing a vehicle chain as a discretized nonlinear flow

Arash Farnam and Alain Sarlette

***Abstract*—This paper deals with the behavior of a chain of vehicles controlling their motion on basis of local relative positions. It is well-known that in such a chain, under any standard linear control (see paper), the deviations of the vehicles from their nominal behavior in reaction to an event at the leading vehicle, grows without bound along the chain as the number of vehicles increases. A saturated linear controller does not appear to solve the issue. We here propose to solve this with a nonlinear controller which ensures that the deviations stay bounded. Our controller design is based on an appropriate discretization of continuous flow equations (partial differential equations) that exhibit stable propagation behavior, namely the Korteweg-de Vries and modified Korteweg-de Vries equations. We propose controllers both for unidirectional links (vehicles only look ahead) and for symmetric bidirectional links, both situations that fail with linear controllers. We argue which mathematical properties motivate these controllers and illustrate the resulting improvement in simulations.**

I. INTRODUCTION

The problem of controlling platoons of vehicles in automated transit systems has been extensively studied during the recent years. Although the physical characteristics vary widely from one system to another, the control problems encountered are similar. It has been shown that grouping vehicles into platoons may provide potential capacity improvements as high as 20 percent, which would also lead to fewer pollution caused by both the reduction of time the vehicles spend on the highway and a low emission intelligent driving model based on smoother speeds. Other benefits include increased safety and reliability, time savings and enhanced related productivity. In order to benefit the most from controlling vehicle platoons, all the technologies must collaborate towards maximum efficiency [1]. Furthermore, the control actions should be reasonable to implement, adaptable to all operational conditions, and simple enough to ensure reliability. Hence most recent works consider acceleration control based on relative measurements and locally distributed decision-making strategies [2], [3], [4].

Consider the problem of one dimensional platooning of N identical vehicles moving on a line, where each of them is separated by a small distance from its front and rear neighbors and required to move in one direction. When even small external disturbances act on the system, the vehicle platoon can undergo strong performance degradation through *string instability*. String stability means, for any perturbation bounded in L_2 norm that is injected at one vehicle, the L_2 norm of the resulting perturbation on the last vehicle N should be bounded independently of N, see [5]. The same authors have shown that essentially, string stability cannot be achieved with linear controllers based on relative measurements and locally distributed decision-making. The string *in*stability of vehicle chains has been further studied for linear control systems [6], [7], [8], [9], [10], [12].

This impossibility to satisfy string instability holds for vehicles modeled as second-order pure integrators. When drag is added the problem gradually disappears. However, as more efficient vehicles should have minimal drag, it seems poorly acceptable to develop future technologies which must rely precisely on drag in order to keep the controlled system string stable. In some systems (space vehicles or the new vacuum tube transit proposal), drag would in fact be absent. Some authors achieve string stability by using absolute velocity in the controllers [16], [12], [13], [14], [15]. Although this essentially corresponds to re-introducing drag, it can be formulated in a smart, possibility optimized way in the framework of passivity-based controllers [15] or of a time headway policy. In the latter, the absolute velocity enters through the time it takes for a following vehicle, at its current speed, to reach the position currently occupied by its predecessor. We here try to avoid relying on this additional knowledge. Another key assumption is that vehicles base their decision on those in front of them. When the control decision is based on distance to one directly preceding vehicle, a quick proof of necessary string instability follows from the Bode integral, as the transfer function from vehicle i to $i+1$ takes the form of a sensitivity function [17]. Allowing coupling with vehicles in front *and behind* oneself, information along the vehicle chain can also travel from the rear to the front and performance must be adapted. Nevertheless, it has been shown that linear controllers also unavoidably imply string instability when vehicles apply a symmetric treatment to one vehicle in front and one vehicle behind them, [11], [18]. We are currently investigating whether a non-trivial variation on the "mistuning-based control" [19], which in simple systems improves the negativity of the eigenvalues from $O(1/N^2)$ to $O(1/N)$, might circumvent string instability.

In this paper however, we propose *nonlinear* unidirectional and bidirectional controllers to guarantee string stability of vehicle platoons. The nonlinear control design is not trivial. Indeed, a nonlinear controller resulting from the saturation of a linear one has been considered in [20] both for the bidirectional and unidirectional cases; and simulations show that the ratio from the input injected at the first vehicle to the motion of the last vehicle keeps growing through the vehicle

Arash Farnam is with the Data Science Lab, Department of Electronics and Information Systems, Faculty of Engineering and Architecture, Ghent University; Technologiepark Zwijnaarde 914, 9052 Zwijnaarde(Ghent), Belgium. Alain Sarlette is with the Data Science Lab, Ghent University, and with the QUANTIC lab, INRIA Paris, 2 rue Simone Iff, 75012 Paris, France. arash.farnam@ugent.be, alain.sarlette@inria.fr.

This paper presents research results of the Belgian Network DYSCO (Dynamical Systems, Control, and Optimization), funded by the Interuniversity Attraction Poles Programme, initiated by the Belgian State, Science Policy Office. A preliminary version of this work has been submitted to the *IFAC-Nolcos workshop 2016*, see [25].

chain, although it does so much more slowly than in the linear case (which is indeed the contribution of [20]). In contrast, we derive nonlinear unidirectional and bidirectional control laws by trying to emulate nonlinear partial differential equations, explicitly the (Modified) Korteweg-de Vries equations, which are third order and provide important examples of a dispersive nonlinear wave process with stable transport phenomena (so-called "soliton solutions"). We apply an appropriate finite difference methods to spatially discretize those partial differential equations towards nonlinear control laws for the discrete chain of vehicles that only use the relative information of neighbor vehicles. Besides a few analysis arguments, simulation results show the ability of our proposed nonlinear controllers to guarantee string stability.

The paper is organized as follows. Section II clarifies the problem setting. Section III presents the (Modified) Korteweg-de Vries equations and the controller design procedure. In Section IV we analyze the resulting vehicle chain behavior with approximation arguments. In Section V we provide further evidence for string stability through simulations.

II. Problem description

Consider N vehicles following each other and modeled as double integrators:

$$\ddot{y}_i = u_i \ \ , \ i = 1, 2, ..., N \tag{1}$$

where y_i is the position of vehicle i and u_i is an acceleration control. The objective of each vehicle is to follow its preceding vehicle at a fixed desired distance d. To achieve this task, it will adapt u_i as a function of observed information. In this paper a main assumption is that this information only conveys the relative situation of consecutive vehicles, i.e. their relative position $y_i - y_{i-1}$ or relative velocity $\dot{y}_i - \dot{y}_{i-1}$. However, we do allow this information to be communicated over a small distance, e.g. vehicle $i+2$ might be aware of the value of $y_i - y_{i-1}$. For simplicity of notation in all the following, we introduce a constant change of variables such that the desired reference is $e_i := y_i - y_{i-1} = 0$ for all i, e.g. in the vehicle chain we make the change of notation $y_i \to y_i - i\,d$.

We consider two scenarios for the flow of information, shown on Fig. 1. In the first scenario, called unidirectional coupling, the control action u_i of vehicle i only depends on relative states of a few preceding vehicles $j < i$. The first few vehicles are considered as "leaders" and do not follow the control laws of the chain. In bidirectional coupling, a vehicle i can react to a few vehicles in front ($j < i$) or behind ($j > i$) itself. The last vehicle is then supposed to react according to the corresponding unidirectional control law, i.e. as if its (absent) follower was perfectly synchronized at distance d.

The objective is more explicitly to guarantee a bounded reaction of all the vehicles to bounded disturbances injected into the system. We here state as formal objective a *string stability* property which is somewhat adapted from the original definition in [5], but which boils down to the same for linear systems. It might also appear closer to practical requirements.

Definition 1: A system parameterized by control parameters c is *string stable* if given any $\epsilon > 0$ there exists a $\sigma > 0$ and a controller tuning c, independent of the number of vehicles N, such that $\|u\|_{L_2} < \sigma$ ensures $\|y_N - y_{N-1}\|_{L_2} < \epsilon$.

For a linear system based on local relative measurements and either unidirectional or bidirectional symmetric communication, it has been proved that it is impossible to achieve string stability [5], [11], [18]. In the nonlinear case, for large N the error $\|y_N - y_{N-1}\|$ might converge to some limit value that depends on c, but not on N nor essentially on the input disturbance σ — think of e.g. a stable limit cycle. If adapting c can scale down the amplitude of that limit arbitrarily, then we would accept the situation as string stable. The purpose of this paper is to show that indeed such designs exist with nonlinear controllers.

III. Nonlinear control design from the Korteweg-de Vries and Modified Korteweg-de Vries partial differential equations

In this section we provide nonlinear unidirectional and bidirectional feedback designs to control the vehicle chain in a "string stable" manner. These designs are inspired by the practical stable behavior of the following nonlinear continuous flow equations (partial differential equations = PDEs). We will hence design the *coupling* among vehicles — unlike the most traditional PDE-based approaches which focus on the design of inputs at the *boundary* of the domain — to mimic microscopic interactions which would give rise to the flows associated to such PDEs. We then expect the vehicle chain to behave like the continuous medium, which is desirable, for all long-range deformations. For too short-range deformations we will have to analyze the system of discrete vehicles as such; the behavior turns out to be satisfactory as well (Section IV-B).

A. The (modified) Korteweg-de Vries PDE

Denote the position along a continuous string by $\eta \in \mathbb{R}$ and time by $\tau \in \mathbb{R}$. The value of some distributed property (e.g. deformation, temperature, strain) along the string is given by a function $v(\eta, \tau)$. Using the typical short notations

$v_\tau(\eta,\tau) = \frac{\partial v(\eta,\tau)}{\partial \tau}$; $v_\eta(\eta,\tau) = \frac{\partial v(\eta,\tau)}{\partial \eta}$; $v_{\eta\eta}(\eta,\tau) = \frac{\partial^2 v(\eta,\tau)}{\partial^2 \eta}$

and so forth, the Korteweg-de Vries (KdV) and modified Korteweg-de Vries (MKdV) equations write, respectively:

$$v_\tau(\eta,\tau) + \beta v(\eta,\tau) v_\eta(\eta,\tau) + \gamma v_{\eta\eta\eta}(\eta,\tau) = 0\ ; \tag{2}$$

$$v_\tau(\eta,\tau) + \beta v^2(\eta,\tau) v_\eta(\eta,\tau) + \gamma v_{\eta\eta\eta}(\eta,\tau) = 0\,. \tag{3}$$

The parameters β and γ are just scaling factors. These equations were first derived in studies of shallow water waves [21].

Authors have characterized so-called *soliton solutions* of the KdV and MKdV equations [23], [24], [26], i.e. perturbations which propagate along the spatial direction without deformation. Indeed (2) and (3) have respectively the solutions:

$$v(z) = \pm \frac{3C}{\beta} \operatorname{sech}^2(3z \frac{\sqrt{C}}{2\sqrt{\gamma}} + \eta_0)\ ; \tag{4}$$

$$v(z) = \pm \sqrt{\frac{6C}{\beta}} \operatorname{sech}(z \frac{\sqrt{C}}{2\sqrt{\gamma}} + \eta_0)\ , \tag{5}$$

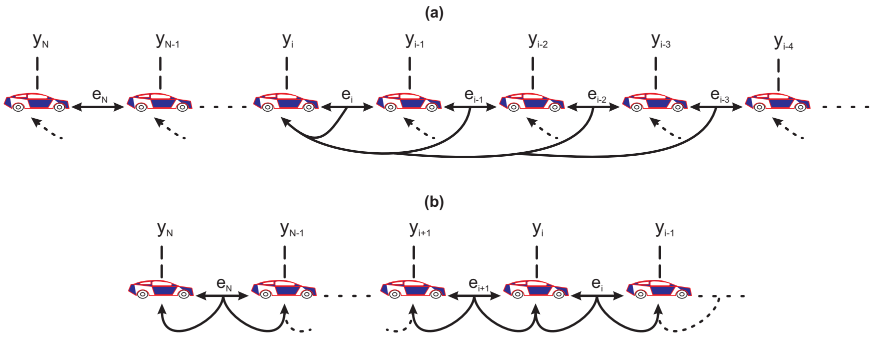


Fig. 1. (a) Unidirectional interconnection scheme. (b) Bidirectional interconnection scheme

where $z = \eta - C\tau$. This expresses that a v-distribution with shape $1/\cosh^2$, resp. $1/\cosh$ (visually similar to a Gaussian) starting around position η_0, preserves its shape and amplitude as it propagates over time τ along the spatial dimension η at velocity C. More "peaked" solitons propagate faster. Moreover, it has been proved [24], [27] that (4) and (5) describe stable solutions of the KdV and MKdV equations, as effects of dispersion and nonlinearity cancel each other.

B. Unidirectional control design procedure

We now explain the procedure of nonlinear control design inspired by KdV and MKdV equations, for the case of a unidirectional vehicle chain (see Fig. 1(a)). I.e., each vehicle i only measures its relative behavior w.r.t. its front vehicle i-1, sends the associated relative position information $e_i = y_{i\text{-}1} - y_i$ to a few following vehicles i+1, ..., i+M and receives the same information $e_k = y_{k\text{-}1} - y_k$ from a few preceding vehicles $k = i\text{-}1, ..., i\text{-}M$. To mimic the nonlinear PDE behavior, we will need $M = 3$.

We start with the KdV equation (2). The procedure, inspired by arguments in [28, p.49-p.50], is similar for the other cases.

To link the KdV equation to the vehicle chain, we must get a second-order time derivative. To this end we make a first substitution $v = y_\eta$, for which (2) implies:

$$y_{\eta\tau}(\eta,\tau) + \beta y_\eta(\eta,\tau) y_{\eta\eta}(\eta,\tau) + \gamma y_{\eta\eta\eta\eta}(\eta,\tau) = 0\,. \qquad (6)$$

We then define a change of variables $\eta = x + h\sqrt{\omega}t$ and $\tau = \frac{h^3}{24\sqrt{\omega}}t$, where the roles of h and ω will be specified later. Then by the chain rule $\frac{\partial^k}{\partial x^k} = \frac{\partial^k}{\partial \eta^k}$, $\frac{\partial}{\partial t} = h\sqrt{\omega}\frac{\partial}{\partial \eta} + \frac{h^3}{12\sqrt{\omega}}\frac{\partial}{\partial \tau}$, and $\frac{\partial^2}{\partial t^2} = h^2\omega\frac{\partial^2}{\partial \eta^2} + \frac{h^4}{12}\frac{\partial^2}{\partial \eta \partial \tau} + (\frac{h^6}{24^2\omega})\frac{\partial^2}{\partial \tau^2}$. To facilitate later expressions we further define $\beta' = \frac{h\beta}{24}$. In these new coordinates, from (6) we get

$$y_{tt} = h^2\omega y_{xx} - 2h^3\beta' y_x y_{xx} - \frac{h^4}{12}\gamma y_{xxxx}\,, \qquad (7)$$

up to a term proportional to $hy_{\tau\tau}$ which we will intentionally keep very small (see analysis section).

We discretize (7) in space with finite differences looking only at vehicles in front. Parameter h is taken as discretization step to map the consecutive vehicles to values of x. *Note thus that the x coordinate characterizes the* index *of the particle in the continuous chain model, in other words its topological place in the chain,* not *its actual position in space; the latter is given by y:*

$$\begin{aligned} y_x &\simeq \frac{y_i - y_{i-1}}{h} \\ y_{xx} &\simeq \frac{(y_i - y_{i-1}) - (y_{i-1} - y_{i-2})}{h^2} - \lambda\frac{h^2}{12}y_{xxxx} \\ y_{xxxx} &\simeq \frac{(y_i - y_{i-1}) - 3(y_{i-1} - y_{i-2})}{h^4} \\ &\quad + \frac{3(y_{i-2} - y_{i-3}) - (y_{i-3} - y_{i-4})}{h^4}\,. \end{aligned} \qquad (8)$$

For $\lambda = 0$ we approximate y_{xx} up to terms of order $h^4 y_{xxxx}$ and this is enough for the nonlinear part of the controller; but for the linear part we must use a more accurate expression including the order $h^4 y_{xxxx}$, which is obtained with $\lambda = 1$.

Substituting (8) into (7) we get:

$$\begin{aligned} \ddot{y}_i &= \tfrac{-(\gamma-11\omega)}{12}(y_i - y_{i-1}) - \tfrac{(3\omega-\gamma)}{4}(y_{i-1} - y_{i-2}) \\ &\quad + \tfrac{(\gamma+\omega)}{12}[-3(y_{i-2} - y_{i-3}) + (y_{i-3} - y_{i-4})] \\ &\quad - 2\beta'\left[(y_i - y_{i-1})^2 - (y_i - y_{i-1})(y_{i-1} - y_{i-2})\right]\,. \end{aligned}$$

The KdV equation features a large invariant space, of all propagating solitons. So to damp deformation modes we add a term proportional to relative velocity and get our final controller design:

$$\begin{aligned} \ddot{y}_i &= \frac{-(\gamma\text{-}11\omega)}{12}(y_i - y_{i\text{-}1}) - \frac{(3\omega\text{-}\gamma)}{4}(y_{i\text{-}1} - y_{i\text{-}2}) \\ &\quad + \frac{(\gamma+\omega)}{12}[-3(y_{i\text{-}2} - y_{i\text{-}3}) + (y_{i\text{-}3} - y_{i\text{-}4})] \\ &\quad - 2\beta'\left[(y_i - y_{i\text{-}1})^2 - (y_i - y_{i\text{-}1})(y_{i\text{-}1} - y_{i\text{-}2})\right] \\ &\quad - b\,(\dot{y}_i - \dot{y}_{i\text{-}1})\,. \end{aligned} \qquad (9)$$

For the MKdV equation, we can make the same substitution $v = y_\eta$ and get:

$$y_{\eta\tau}(\eta,\tau) + \beta y_\eta^2(\eta,\tau) y_{\eta\eta}(\eta,\tau) + \gamma y_{\eta\eta\eta\eta}(\eta,\tau) = 0\,. \qquad (10)$$

Using the same change of variables and with $\beta' = \frac{\beta}{36}$, we have

$$y_{tt} = h^2\omega y_{xx} - 3h^4\beta' y_x^2 y_{xx} - \frac{h^4}{12}\gamma y_{xxxx} \qquad (11)$$

up to a small term proportional to $hy_{\tau\tau}$. Finally with the same discretization method only looking vehicles at front, we obtain after adding the velocity damping term:

$$\begin{aligned}\ddot{y}_i &= \tfrac{-(\gamma\text{-}11\omega)}{12}(y_i - y_{i\text{-}1}) - \tfrac{(3\omega\text{-}\gamma)}{4}(y_{i\text{-}1} - y_{i\text{-}2}) \\ &+ \frac{(\gamma+\omega)}{12}[-3(y_{i\text{-}2} - y_{i\text{-}3}) + (y_{i\text{-}3} - y_{i\text{-}4})] \\ &-3\beta'\left[(y_i - y_{i\text{-}1})^3 - (y_i - y_{i\text{-}1})^2(y_{i\text{-}1} - y_{i\text{-}2})\right] \\ &-b(\dot{y}_i - \dot{y}_{i\text{-}1})\,.\end{aligned} \tag{12}$$

C. Bidirectional control design procedure

We now design a bidirectional controller where the feedback action of each vehicle i is based on measurements of its relative position with respect to its directly preceding $(i-1)$ and directly following $(i+1)$ vehicles. Hence in principle no communication is required. However, this controller relies on the fact that (induced) events at the back of the chain influence what happens at the front, while for a moving chain of vehicles it may be more desirable to isolate the front from what happens behind. Also, in this particular structure, we are not free to tune all the parameters of the PDE independently. This is not surprising since a 4th order derivative (see below) cannot be approximated, independently, with only 3 values (namely at $i, i-1, i+1$). Thus the communication-free method works only if we are happy with the resulting particular tuning values.

The design proceeds as for the unidirectional case, just with a different discretization and change of variables.

We start again with the KdV equation (2). We then define $\eta = x - h\sqrt{\omega}t$ and $\tau = \frac{h^3}{24\sqrt{\omega}}t$. By the chain rule $\frac{\partial^k}{\partial x^k} = \frac{\partial^k}{\partial \eta^k}$, $\frac{\partial}{\partial t} = -h\sqrt{\omega}\frac{\partial}{\partial \eta} + \frac{h^3}{12\sqrt{\omega}}\frac{\partial}{\partial \tau}$, and $\frac{\partial^2}{\partial t^2} = h^2\omega\frac{\partial^2}{\partial \eta^2} - \frac{h^4}{12}\frac{\partial^2}{\partial \eta \tau} + (\frac{h^6}{24^2\omega})\frac{\partial^2}{\partial \tau^2}$. With $\beta' = \frac{h\beta}{24}$, equation (6) becomes

$$y_{tt} = h^2\omega y_{xx} + 2h^3\beta' y_x y_{xx} + \tfrac{h^4}{12}\gamma\, y_{xxxx}\,, \tag{13}$$

again up to a term proportional to $hy_{\tau\tau}$ which we will intentionally keep very small (see analysis section).

We discretize the 1st order derivative as usual, but we merge the 2nd and 4th order derivatives. Indeed, writing:

$$\begin{aligned} y_x &\simeq \frac{y_i - y_{i-1}}{h} = \frac{y_{i+1} - y_i}{h} \\ y_{xx} &\simeq \frac{y_{i+1} - 2y_i + y_{i-1}}{h^2} - \frac{h^2}{12}y_{xxxx}\,, \end{aligned} \tag{14}$$

we see that the second expression directly covers two terms of (13), provided $\omega = \gamma$. Regarding the y_{xx} appearing in the nonlinear term, like for the unidirectional controller, it is multiplied by small enough factors such that the y_{xxxx} is of higher order and can be dropped. Thus the two elements in (14) are enough for us to discretize (13) for $\omega = \gamma$. Then substituting (14) into (13) and adding as before a term proportional to relative velocity, we get:

$$\begin{aligned}\ddot{y}_i &= -\gamma[(y_i - y_{i+1}) + (y_i - y_{i-1})] \\ &-\beta'[(y_i - y_{i-1})^2 - (y_{i+1} - y_i)^2] \\ &-b(\dot{y}_i - \dot{y}_{i-1})\,.\end{aligned} \tag{15}$$

If we want $\omega \neq \gamma$, then a different discretization has to be used, looking at two vehicles in front and two behind to cover the y_{xxxx} term in (13) with the proper coefficient.

The MKdV equation works very similarly, using the same change of variables and finite differences, but just defining $\beta' = \frac{\beta}{36}$. This yields respectively the PDE and the final controller for $\omega = \gamma$:

$$y_{tt} = h^2\omega y_{xx} + 3h^4\beta' y_x^2 y_{xx} + \frac{h^4}{12}\gamma y_{xxxx}\,, \tag{16}$$

$$\begin{aligned}\ddot{y}_i &= -\gamma[(y_i - y_{i+1}) + (y_i - y_{i-1})] \\ &-\beta'[(y_i - y_{i-1})^3 - (y_{i+1} - y_i)^3] \\ &-b(\dot{y}_i - \dot{y}_{i-1})\,.\end{aligned} \tag{17}$$

IV. Analysis of the resulting vehicle chain

In this section we provide a partial analysis of the systems described by (9), (12), (15) and (17). We argue that for long-range and slowly propagating modes, the system is string stable by approximating the transport behavior of the KdV and MKdV equations. On the other hand, short-range and fast varying disturbance signals usually have a smaller amplitude (to have a similar energy), therefore their effect is analyzed on the linearized system. Those arguments do not constitute a full proof (yet) of string stability of our system, but in conjunction with the simulation results of Section V, these are good indications that the KdV and MKdV approximation approach improves on all linear controllers.

A first important observation is that the nonlinear equations feature a natural rescaling. Indeed, for a given value $\beta' = \bar{\beta}$ denote $\bar{y}_1(t), ..., \bar{y}_N(t)$ a solution of the equations describing one of the controlled vehicle chains, with arbitrary additive noise $\bar{u}_1(t), ..., \bar{u}_N(t)$ acting as inputs to the vehicles. Then it is obvious that $y_1(t) = \bar{y}_1(t)/\alpha$, ..., $y_N(t) = \bar{y}_N(t)/\alpha$ is a solution of the same equations, under additive noises $u_1(t) = \bar{u}/\alpha_1(t)$, ..., $u_N(t) = \bar{u}_N(t)/\alpha$, if we use the different value $\beta' = \alpha\bar{\beta}$ in case of (9) or (15) (KdV-based) and if we use $\beta' = \alpha^2\bar{\beta}$ in case of (12) or (17) (MKdV-based). This just expresses that as the vehicle chain becomes more and more nonlinear (larger β'), the same qualitative behavior will appear at smaller and smaller disturbance amplitudes. *This relation is exact* and may inform the tuning of the controller. In particular, it reduces our task to showing that for some controller tuning and disturbance scale, the error on vehicle N can be bounded by some $\bar{\epsilon} > 0$ independently of N. Then indeed just by rescaling β', the resulting nonlinear controller can be ensured to comply with any requested $\epsilon > 0$ in Definition 1.

A. Slowly varying perturbations

The objective of this section is to argue how the controlled vehicle chains governed by (9),(15), (12) or (17), are good approximations of the KdV and MKdV equations respectively, at least for slowly varying boundary conditions i.e. slow variations in the motion of the leading vehicles. We then hope that such perturbations propagate along the vehicle chain without amplification. Indeed, the stable, "pure transport" behavior of the KdV and MKdV equations were the initial motivation for our controller design.

Concretely: typical solutions of the Korteweg-de Vries PDEs take the form of solitons (4) or (5), which propagate without damping nor amplification [24]. Our addition of velocity damping suggests that we would rather have damped solitons, if things go well. Towards this last hypothesis, it has been proved [23], [27] that (4) and (5) describe *stable* solutions of the KdV and MKdV equations respectively, as effects of dispersion and nonlinearity, which precisely cancel each other on the soliton, in fact push the system back towards it when departing from that shape. This would be a string stable behavior for the vehicle chain, just evacuating the disturbance towards its boundary (and damping it, as we add $b > 0$).

We will therefore here check what happens when the vehicle chain approximately follows a soliton solution. More precisely, under that assumption, we will check that the perturbations induced into the original KdV or MKdV equations by our discretizing approximations, remain small. Such perturbations would indeed be stably rejected by the PDE, hence as usual in the discretization literature, one might hope that the discretized vehicle chain system will indeed be drawn towards the soliton solutions. This is true because we have checked by linearization that the discretized system, i.e. the actual controlled vehicle chain, is indeed a stable system for input y_1 (although maybe not string stable, i.e. the transfer function to y_N is bounded but not independently of N).

We first check the transition from e.g. (6) to (7), where we neglected a term in $y_{\tau\tau}$. Since the soliton solution is a function of $z = \eta - C\tau$, the order of magnitude of $y_{\tau\tau}$ that appears in the change of variables is C times the order of magnitude of $y_{\tau\eta}$. From this, the term in $y_{\tau\tau}$ can be viewed as a small perturbation with respect to the KdV and MKdV term in $y_{\tau\eta}$, provided Ch^2 is small.

We next check the discretization approximation. The full explanation is a bit technical and deferred to the Appendix.

- First note: the objective is to show that when the leader motion (or other input disturbance) is compatible with a soliton, then the vehicles in the vehicle chain will have a motion close to the PDE's soliton solution *sampled at discrete steps.* The change of variables for unidirectional and bidirectional controllers respectively yield

$$\begin{aligned} z &= x + (h\sqrt{\omega} - Ch^3/(24\sqrt{\omega}))t \quad \text{and} \\ z &= x - (h\sqrt{\omega} - Ch^3/(24\sqrt{\omega}))t \ \ \text{with } \omega = \gamma, \end{aligned}$$

  where individual vehicle indices $i = 0, 1, 2, ...$ correspond to sampling at $x = 0, h, 2h, ...$
- A first condition (see appendix) is that we must select h small enough. Hence we retrieve the intuitive condition of good discretization, namely that a characteristic length of the soliton should cover a large number of vehicles.
- A second condition (see appendix) is that C must be small enough. This means slow velocity of the PDE soliton. The velocity of propagation *between vehicles* however, as h is also small, is computed as $|(\sqrt{\omega} - Ch^2/(24\sqrt{\omega}))|$ hence dominated by $\sqrt{\omega}$, where $\omega = \gamma$ in the bidirectional case. Thus the perturbation does not necessarily propagate slowly among the vehicles. Nevertheless the relative variation of $y_i(t)$ over time, at a given vehicle i, remains small: this is just because the soliton covers many vehicles, so moving the soliton center from i to $i+1$ implies little change in y_i.
- The first condition also has an indirected effect, as β' the control parameter and β in the soliton amplitude are related by h. Together with the condition on C, this implies that for given β' only solitons of sufficiently small amplitude will be well approximated. Such bound is also standard in discretization schemes. For smaller β', i.e. a closer to linear system, the KdV-like propagation of larger solitons will be well approximated.

The formal conclusion can be summarized as follows. For given β', γ, ω in the vehicle chains (9), (12), (15) or (17), there exist sufficiently small $\bar{h} > 0$ and $\bar{C} > 0$ such that: If the motion of the leading vehicles is compatible with a succession of soliton solutions (4) (resp. (5)) with $|C| < \bar{C}$ and with $\beta > 24\beta'/\bar{h}$ (resp. $\beta > 36\beta'$), then we can expect the vehicle chain with $b = 0$ to move similarly to the solutions implied by sampling the KdV and MKdV solitons at discrete points in space. The practical implication – see simulations below – is that the nonlinearly coupled vehicles can avoid string instability for slow and long-range perturbations, which are usually the worst ones for linear controllers.

Remark: A surprising effect of the change of variables is that for the unidirectional controller, the well-approximated solitons would propagate from back to front of the vehicle chain – i.e. the rear vehicles appear to be anticipating the soliton before the front vehicles actually feature it. This is confirmed in simulations, see Fig. 2. Correctly predicting solitons, and *only* solitons, on the basis of a small interval of the leaders' trajectory, might seem counterintuitive and fragile but it is certainly not logically impossible. The bidirectional controllers do not feature this particularity.

B. Linear Stability Analysis

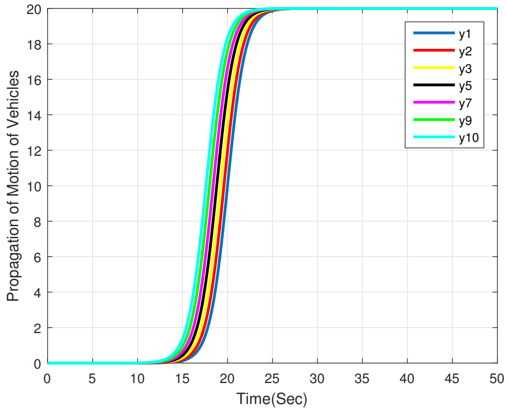
The effect of higher-frequency disturbances, superimposed on these long-range soliton-type inputs, can be investigated by linearization, assuming that they would typically feature small amplitudes.

For the unidirectional controllers, linearizing with $\delta e_i = e_i - \bar{e}_i$ around $\bar{e}_i$, approximating $a_i := \bar{e}_i - \bar{e}_{i-1}$ as a static situation, we get in Laplace domain

$$\delta e_i(s) = \sum_{k=1}^{4} T_k(s)\ \delta e_{i-k}(s)\ ,$$

where for (9) and (12) we have respectively:

$$\begin{aligned} T_k(s) &= \frac{Q_k(s)}{s^2 + bs + (\frac{(\gamma - 11\omega)}{12} + 2\beta'(2a_i - a_{i\text{-}1}))} \quad (18) \\ T_k(s) &= \frac{Q_k(s)}{s^2 + bs + (\frac{(\gamma - 11\omega)}{12} + 3\beta'(3a_i^2 - 2a_i a_{i\text{-}1}))} \end{aligned}$$



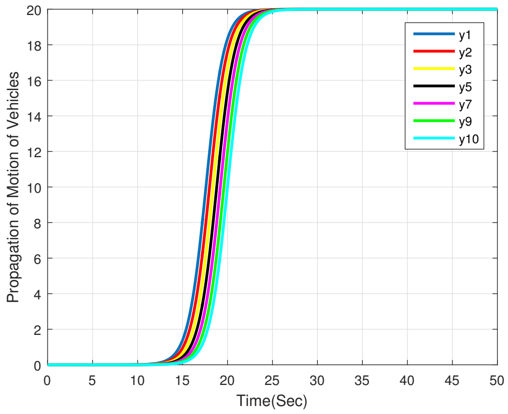


Fig. 2. (top) Motion of ten following vehicles under bidirectional control law (9) with $b = 0$, when the leading vehicle follows an integrated soliton solution, which takes the form $atanh(\cdot)$. The soliton appears to move forward as the nonlinear reaction of followers transforms ever tinier soliton-compatible fluctuations of their leader into solitons that switch ever earlier. (bottom) Motion of ten following vehicles under unidirectional control law (15) with $b = 0$, when the leading vehicles follow an integrated soliton solution, which takes the form $atanh(\cdot)$. The soliton appears to move backward.

and $Q_k(s)$ are a first-order polynomial for $k = 1$ and some constants for $k > 1$. For the bidirectional controllers, we have

$$\delta e_i(s) = T_1(s)\ \delta e_{i-1}(s) + T_2(s)\ \delta e_{i+1}(s)$$

where for (15) and (17) respectively:

$$\begin{aligned} T_k(s) &= \frac{Q_k(s)}{s^2 + bs + 2(\gamma + \beta'(a_i - b_i))} \qquad (19) \\ T_k(s) &= \frac{Q_k(s)}{s^2 + bs + 2(\gamma + 3/2\beta'(a_i^2 - b_i^2))} \end{aligned}$$

and the $Q_k(s)$ are first-order polynomials in s.

These four linear models are composed of stable transfer functions, provided $b \geq 0$ and additional conditions respectively $\frac{\gamma-11\omega}{24\beta'} > (a_{i-1} - a_i)$, $\frac{\gamma-11\omega}{36\beta'} > (2a_i a_{i-1} - 3a_i^2)$, $\frac{\gamma}{\beta'} > (b_i - a_i)$ and $\frac{\gamma}{\beta'} > (b_i^2 - a_i^2)$ which relate our controllers' tuning and the expected state around which we linearize the vehicle chain. Explicitly, given a typical bound on low-frequency tracking errors, one would satisfy the conditions for stability of the linearized system by choosing β' not too large and $\gamma > 0$ large enough.

For high enough frequencies, the transfer functions in (18) and (19) all have a small modulus, implying string stability, like in most linear systems. For low frequencies, we hope to observe a soliton-like behavior. The domain of medium-frequency and medium-amplitude perturbations remains open, as we were not able to mathematically characterize the non-linear vehicle chain system in this range. We next present simulations which show that apparently, string stability indeed holds with our controllers.

V. Simulations

The main goal of the simulations is to show that our nonlinear unidirectional control laws (9) and (12) looking only at four vehicles in front and bidirectional controllers (15) and (17) looking only at one vehicle in front and one behind, lead to bounded disturbances as the number of vehicles in the chain grows. For reference, we also compare the results to simulations with other controllers. The first ones are simple linear controllers, unidirectional and symmetric bidirectional respectively:

$$\begin{aligned} \ddot{y}_i &= -k(y_i\text{-}y_{i\text{-}1}) - b\,(\dot{y}_i\text{-}\dot{y}_{i\text{-}1})\,, \qquad (20) \\ \ddot{y}_i &= -k(y_i\text{-}y_{i\text{-}1} + y_i\text{-}y_{i+1}) - b\,(\dot{y}_i\text{-}\dot{y}_{i\text{-}1} + \dot{y}_i\text{-}\dot{y}_{i+1})\,, \end{aligned}$$

with some $k, b > 0$. The second ones are saturated versions of such linear controllers, as studied in [20]. Recall that *any* linear controller — unidirectional looking at an *arbitrary* fixed number of preceding vehicles, or symmetric bidirectional looking at one vehicle in front and one behind — is provably string unstable; this is also what simulations will show with (20). The saturation-based controller would be an easy attempt to bound the vehicle chain's reaction to disturbances, although [20] has not investigated it in this sense. Our simulations will show however that this saturation is apparently not enough to ensure string stability in a second-order system.

In contrast, our PDE-inspired controllers all seem to be able to ensure string stability. This conclusion is of course based on a finite sample of simulations, which is not a proof for a nonlinear system. Moreover, we cannot verify the definition of string stability but only check how a necessarily finite signal seems to get amplified along a necessarily finite chain. However, theses conclusions should be relevant for all practical purposes. After all, the linear and saturation-based controllers are all "robustly string unstable", as a component that is amplified along the chain emerges for random inputs.

The first simulation results, see Figure 3, show typical system trajectories obtained for a random input at the leading vehicles. Visually, with our 4 controllers, the error on all vehicles up to the last one (here $N = 50$) are amplified by a bounded factor ($\sim$2) with respect to the first one – i.e. we seem to have string stability. The reader may want to track how some signal features propagate along the chain. Remember that our communication-less bidirectional coupling leaves less choice in the controller tuning. Larger values of N and random inputs with other frequency spectra (not shown) feature qualitatively similar behavior.

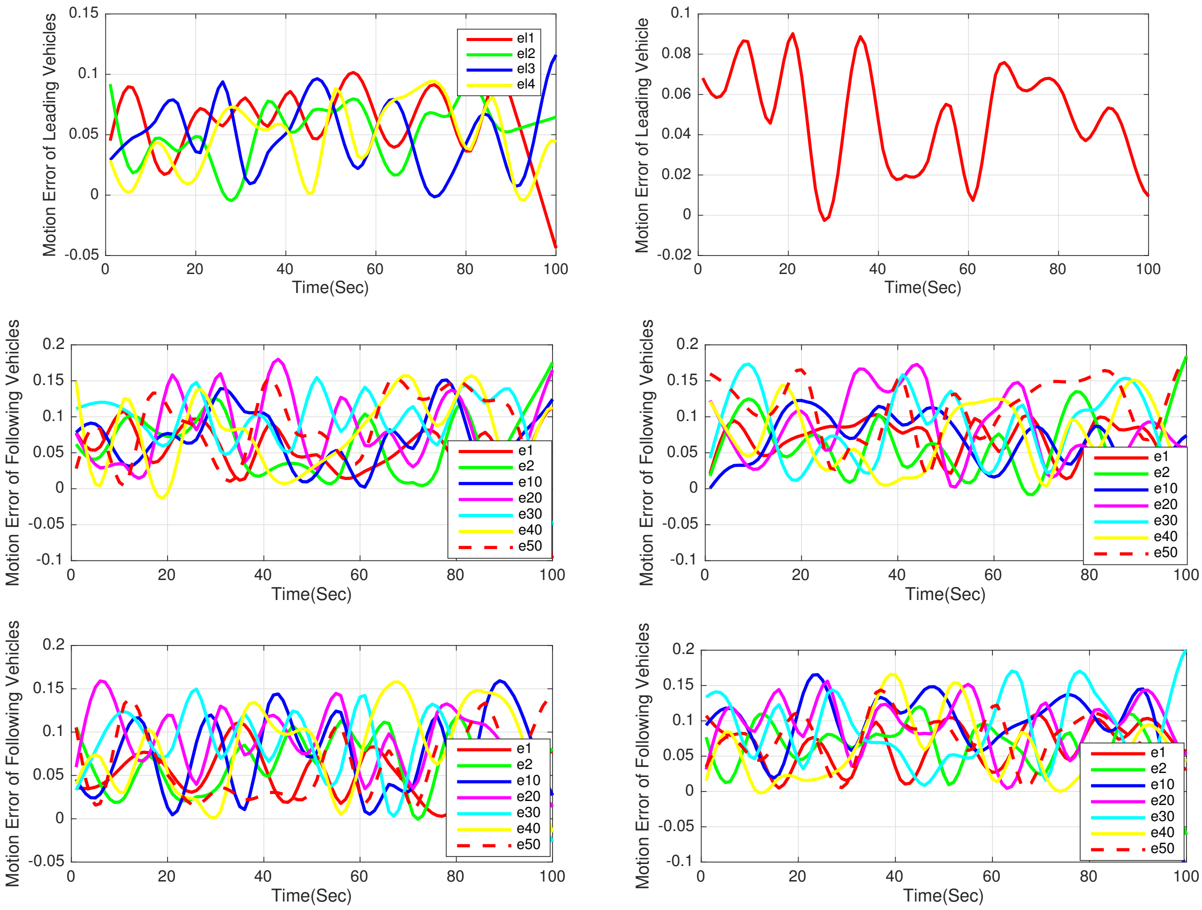


Fig. 3. Behavior of our PDE-inspired control laws under random motions of the leading vehicles. The control parameters are chosen to be $\gamma = 200$, $\omega = 10$ for unidirectional, $\beta' = 80$, $b = 1$. The left plots consider unidirectional coupling: 4 leaders' inputs on top row, KdV-based followers (9) on middle row, MKdV-based followers (12) on lower row. The right plots concern bidirectional coupling: 1 leader input on top row, (15) on middle row, (17) on lower row. In all the cases, the followers' deviations does not increase along the vehicle chain.

We next investigate the effect of disturbance amplitude and of β'. For better visualization, we do not show sets of trajectories anymore, but instead we estimate by MonteCarlo simulations the standard deviation of the last vehicle $\sqrt{E(e_N^2(T))}$, for T a sufficiently large time such that transients die out (here $T = 4000$s) and the expectation taken over leading vehicle inputs. Figure 4(top) shows how this indicator evolves with N, for the bidirectional controller (15). Similar results are obtained with the other controllers. *A very nice feature appears, which we anticipated but did not rigorously prove: the disturbance on the last vehicle N convergences to a bounded value, independent of N and of the input disturbance amplitude.* If we want $\sqrt{E(e_N^2(T))}$ to be α times smaller, according to the rescaling mentioned at the beginning of Section IV, we need both to divide the input amplitudes by α and to multiply β' by α. This prediction is confirmed on the bottom figure with $\alpha = 5$.

Finally, we compare the performance of our different controllers with the linear and saturation-based controllers. We quantify the effect of random disturbances by using like in [20] the expected first-to-last ratio for random inputs:

$$R_{FTL} := \frac{\sqrt{E(e_N^2(T))}}{\delta_0}\,, \tag{21}$$

where δ_0 is the L_2 norm of the random input. In the linear case, R_{FTL} is exactly the H_2 norm of the transfer function from the random motion of first vehicle to the position tracking error of last vehicle. Figure 5 shows R_{FTL} versus number of vehicles N for a fixed $\delta_0 = 0.5$. For all our nonlinear controllers (9), (12), (15) and (17), the ratio R_{FTL} appears to converge to a constant value, hinting at string stability, while in contrast both for the linear controllers (20) and for the saturation-based nonlinear controllers of [20] the ratio keeps growing with N.

This behavior can also be observed on single trajectories, as illustrated e.g. on Fig. 6 with the step response. With our nonlinear controller the edge of the step is damped, while with the linear controller it rather leads to amplified peaks along the chain. Similar damping behavior is observed with other PDE-inspired controllers, and similar amplified peaks with other linear or saturation-based controllers (not shown). The peaks can be understood similarly to the Gibbs phenomenon: as a few low frequencies get amplified by the linear controller while the others do not, the behavior towards the end of the

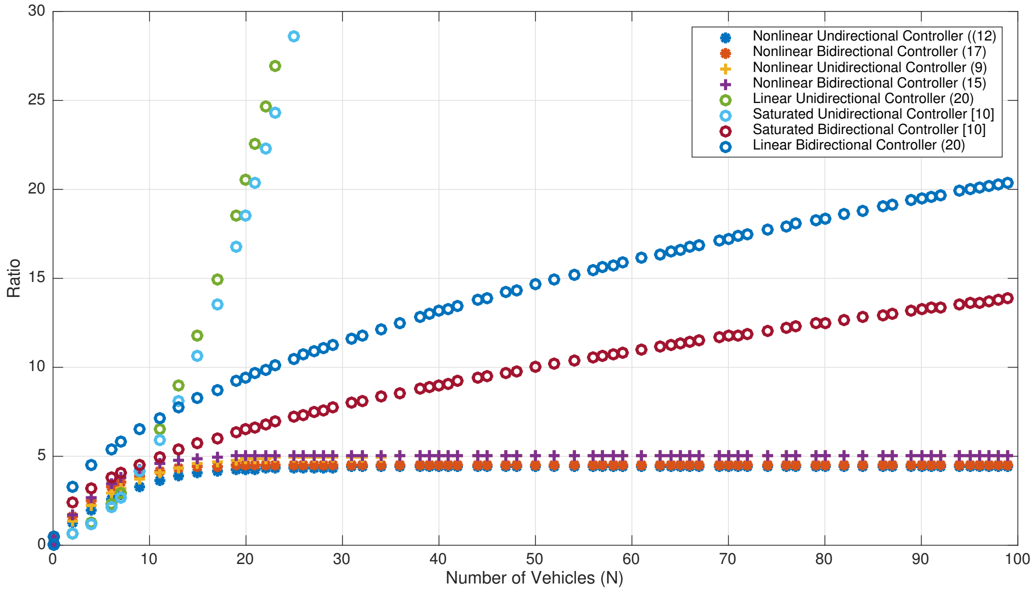


Fig. 5. First-to-last perturbation ratio R_{FTL} (see text) as a function of length of the vehicle chain. For all our PDE-inspired controllers, with the same parameters as Fig. 3, it appears to converge to a constant value, hinting at string stability (one curve is just a rescaling of Fig.4(top)). In comparison, for the linear controller with $b = 1$, $k = 50$ and for the saturated controller, the perturbation ratio keeps growing with the chain.

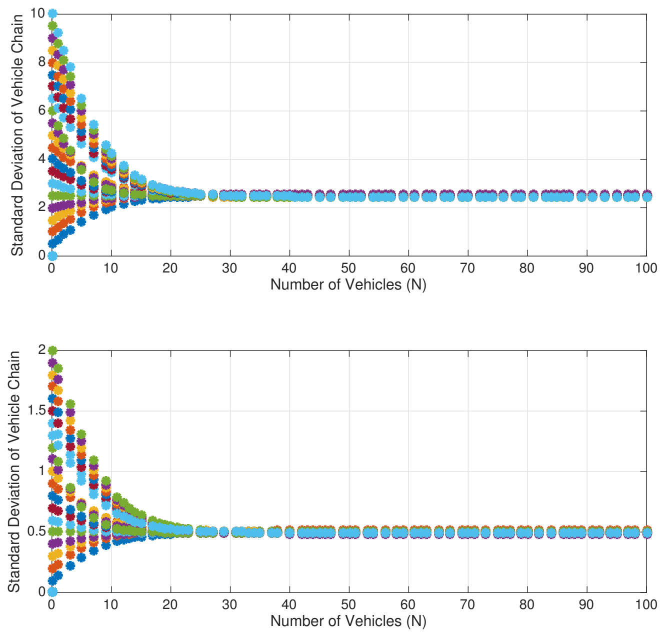


Fig. 4. Standard deviation of the last vehicle $\sqrt{E(e_N^2(T))}$, as a function of the length of the vehicle chain, for random inputs of different amplitudes. The controller is (15) with the same tuning as for Fig. 3 on the top figure, in particular $\beta' = 80$; and with modified $\beta' = 400$ on the bottom figure. The key nice feature is that the value of $\sqrt{E(e_N^2(T))}$ converges to a bound independent of N and of the input amplitude. Multiplying β' by 5 divides this bound by 5, as predicted by the theory.

chain gets closer and closer to the truncated Fourier series of the input signal's edges.

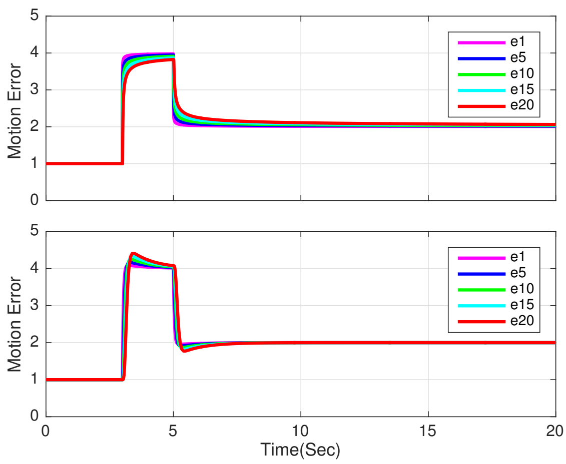


Fig. 6. Reaction to a two-step input at the leader. Motion error of following vehicles with (top) our nonlinear unidirectional control law and (bottom) the linear unidirectional control law (20).

VI. Conclusion

We have proposed a nonlinear controller for a chain of vehicles with double-integrator dynamics inspired by the Korteweg-de Vries nonlinear partial differential equation. The vehicles are controlled based only on relative position measurements between consecutive vehicles. We have obtained two schemes, depending on the discretization used to relate the PDE to the discrete chain of vehicles. In the first one, each vehicle only reacts to those in front, and communication of measurement values from three preceding vehicles is needed. In the second one, each vehicle reacts to errors with respect to the directly preceding and the directly following vehicle, and no communication is required. We have shown, through simulations, that these controllers appear to ensure *string stability*, i.e. a perturbation is *not* amplified indefinitely along an infinite chain. This is a striking improvement with respect to linear controllers — for which it has been proved that string stability cannot be achieved — and also with respect to a linear controller with saturation. We give some mathematical arguments as to why this string stability might hold, although currently a complete proof is still lacking.

From a more general perspective, this paper shows that PDE models, which are widely used to model flows of large numbers of vehicles, can also be used to inform the design of control laws that couple vehicle behaviors at the microscopic scale. Such approach has already been proposed for linear systems [19], but its power for control design could be more striking in the nonlinear context. We have here focused on a particular control-theoretic objective. The positive results could motivate this approach towards mimicking other desirable flow features.

REFERENCES

[1] D. Gillen, E. Chang, and D. Johnson "Productivity benefits and cost efficiencies from its applications to public transit: The evaluation of avl". *California PATH Working Paper UCB-ITS-PWP-2000-16, University of California*, Berkeley, 2000.

[2] S. Darbha and K. R. Rajagopal, "Intelligent cruise control systems and traffic flow stability," *Transportation Research Part B*, Vol. 7, pp. 329-352, 1999.

[3] P. Ioannou, C. Chien, "Autonomous intelligent cruise control", *IEEE Trans. on Vehicular Technology* Vol. 42, pp. 657-672, 1993.

[4] M. Papageorgiou, "Application of automatic control to traffic flow problems" *Lecture Notes in Control and Information Sciences*, vol. 50, Springer, Berlin, 1983.

[5] D. Swaroop, "String stability of interconnected systems: An application to platooning in automated highway systems," *PhD thesis, University of California, Berkeley*, 1994.

[6] K. C. Chu, "Decentralized control of high-speed vehicular strings," *Transportation Science*, vol. 8, no. 4, pp. 361-384, 1974.

[7] S. Klinge, "Stability issues in distributed systems of vehicle platoons," Master's thesis, Otte-von-Guericke-University Magdeburg, *http:/www.hamilton.ie/publications.htm*, 2008.

[8] S. Sheikholeslam and C. Desoer, "Longitudinal control of a platoon of vehicles," *Proc. American Control Conf.*, pp. 291-297, 1990.

[9] W. Levine and M. Athans, "On the optimal error regulation of a string of moving vehicles," *IEEE Trans. Automatic Control*, vol. 11, no. 3, pp. 355-361, 1966.

[10] D. Swaroop and J. Hedrick, "String stability of interconnected systems," *IEEE Trans. Automatic Control*, vol. 41, no. 3, pp. 349-357, 1996.

[11] P. Seiler, A. Pant, and K. Hedrick, "Disturbance Propagation in Vehicle Strings," *IEEE Trans. Automatic Control*, vol. 49, no. 10, pp. 1835-1842, 2004.

[12] J. Ploeg, D.P. Shukla, N. van de Wouw and H. Nijmeijer, "Controller synthesis for string stability of vehicle platoons," *IEEE Trans.Intell.Transp. Systems*, vol. 15, no. 2, pp.854-865, 2014.

[13] C. Chien and P. Ioannou, "Automatic Vehicle following," *Proc. American Control Conf.*, pp. 1748-1752, 1992.

[14] S. Klinge, and R. H. Middleton, "Time headway requirements for string stability of homogenous linear unidirectionally connected systems," *Proc. IEEE Conf. on Decision and Control*, pp. 1992-1997, 2009.

[15] S. Knorn , A. Donaire, J. C. Aguero b and R. H. Middleton, "Passivity-based Control for Multi-Vehicle Systems Subject to String Constraints," *Automatica*, vol. 50, pp. 3224-3230, 2014.

[16] J. Rogge and D. Aeyels(2008), Vehicle platoons through ring coupling, *IEEE Transactions on Automatic Control*, Vol. 53(6), pp.1370-1377.

[17] K.J. Aström and R.M. Murray, *Feedback systems: an introduction for scientists and engineers*, Princeton University Press, 2010.

[18] P. Barooah and J. P. Hespanha, "Error amplification and disturbance propagation in vehicle strings with decentralized linear control," *Proc. IEEE Conf. on Decision and Control*, pp. 4964-4969, 2005.

[19] P. Barooah, P. G. Mehta and J. P. Hespanha, "Mistuning-based control design to improve closed-loop stability of vehicular platoons," *IEEE Trans. Automatic Control*, vol. 54, no. 9, pp. 2100-2113, 2009.

[20] H. Hao and P. Barooah "Stability and robustness of large platoons of vehicles with double-integrator models and nearest neighbor interaction " *International Journal of Robust and Nonlinear Control*, vol. 23, pp. 2097-2122, 2013.

[21] D. J. Korteweg and G. De Vries, "On the change of form of long waves advancing in a rectangular channel, and on a new type of long stationary waves," *Philosophical Magazine*, vol. 5, no. 39, pp. 422-443, 1895.

[22] J. Monteil, R. Billot, J. Sau, and N. E. El Faouzi "Linear and Weakly Nonlinear Stability Analyses of Cooperative Car-Following "*IEEE Trans. Intelligent Transportation Systems*, vol. 15, no. 5, pp. 2001-2013, 2014.

[23] H. J. S. Dorren and R. Snieder, "A stability analysis for the Korteweg-de Vries equation," *http://arxiv.org/abs/solv-int/9605005*, 1996.

[24] H. J. S. Dorren, "Singular and unstable solutions of the Korteweg-de Vries hierarchy," *Journal of Mathematical Physics*, vol. 37, 1996.

[25] A. Farnam and A. Sarlette, "Achieving string stability with nonlinear control inspired by a PDE", submitted to Proc. 10th IFAC symposium on Nonlinear Control Systems, Monterrey CA, August 2016.

[26] N. F. Smyth and T. A. L. Worthy, "Solitary wave evolution for mKdV equations," *Wave Motion*, vol. 21, no. 3, pp. -263-275, 1995.

[27] S. Hakkaev, I. D. Iliev and Ki. Kirchev, "Stability of periodic traveling waves for complex modified Korteweg-de Vries equation," *Journal of Differential Equations*, vol. 284, no. 10, pp. 2608-2627, 2010.

[28] R.S. Palais, "An introduction to wave equations and solitons," *The Morningside Center of Mathematics, Chinese Academy of Sciences, Beijing*, 2000.

APPENDIX

In the following, we investigate under which conditions the nonlinear controllers (9), (15) and (12), (17), with $b = 0$, are good approximations of (22) and (23), respectively.

A. Controllers Derived from the KdV equation

To investigate the approximation, we go back from (9) and (15) to the PDE but keeping all error estimations in the Taylor developments. The discretizations yield respectively:

$$\begin{aligned} y_{tt} &= \omega(h^2 y_{xx} + \tfrac{h^4}{12} y_{xxxx} + \tfrac{h^6}{360} y_{xxxxxx} + O(h^8)) \\ &\quad -(\gamma+\omega)(\tfrac{h^4}{12} y_{xxxx} + \tfrac{h^6}{360} y_{xxxxxx} + O(h^8)) \\ &\quad -\beta'(2h^3 y_x y_{xx} + \tfrac{h^5}{6} y_x y_{xxxx} + \tfrac{h^5}{3} y_{xx} y_{xxx} + O(h^7)) \end{aligned} \tag{22}$$

$$\begin{aligned} y_{tt} &= \gamma(h^2 y_{xx} + \tfrac{h^4}{12} y_{xxxx} + \tfrac{h^6}{360} y_{xxxxxx} + O(h^8)) \\ &\quad +\beta'(2h^3 y_x y_{xx} + \tfrac{h^5}{6} y_x y_{xxxx} + \tfrac{h^5}{3} y_{xx} y_{xxx} + O(h^7)) \end{aligned} \tag{23}$$

which a priori are relevant *if we select a small value for h, compared to the characteristic length of the solutions $y(x,t)$ of the PDE corresponding to a particular boundary condition.* The leading vehicles must induce slow perturbations in this sense.

Working back the change of variables $(\eta,\tau) \leftrightarrow (x,t)$, equations (22) and (23) become respectively:

$$\begin{aligned} 0 \;=\; & y_{\tau\eta} + \tfrac{24\beta'}{h} y_\eta y_{\eta\eta} + \gamma y_{\eta\eta\eta\eta} \\ & +2h\beta' y_\eta y_{\eta\eta\eta\eta} + 4h\beta' y_{\eta\eta} y_{\eta\eta\eta} \\ & +\tfrac{h^2}{30}\gamma y_{\eta\eta\eta\eta\eta\eta} + \tfrac{h^2}{48\omega} y_{\tau\tau} + O(h^3)\,; \end{aligned} \tag{24}$$

$$
\begin{aligned}
0 \;=\;& y_{\tau\eta} + \tfrac{24\beta'}{h} y_\eta y_{\eta\eta} + \gamma y_{\eta\eta\eta\eta} \\
& + 2h\beta' y_\eta y_{\eta\eta\eta\eta} + 4h\beta' y_{\eta\eta} y_{\eta\eta\eta} \\
& + \tfrac{h^2}{30}\gamma y_{\eta\eta\eta\eta\eta\eta} - \tfrac{h^2}{48\gamma} y_{\tau\tau} + O(h^3)\,.
\end{aligned} \tag{25}
$$

The first line of (24) and (25) represents the KdV equation with $v = y_\eta$. Hence we want to show that under some conditions, the other terms can be seen as small perturbations.

We will therefore assume that the systems follow a soliton solution of the KdV equation. Since the soliton solution is a function of $z = \eta - C\tau$, the order of magnitude of $y_{\tau\tau}$ is C times the order of magnitude of $y_{\tau\eta}$. Hence the terms $\frac{h^2}{48\omega} y_{\tau\tau}$ and $\frac{h^2}{48\gamma} y_{\tau\tau}$ above can be viewed as a small perturbations with respect to the KdV term in $y_{\tau\eta}$ provided Ch^2 is small. Similarly, plugging in the solution (4), we can establish that the order of magnitude of (X_1) is (X_2) times the magnitude of (X_3), with respectively

$$
\begin{aligned}
((X_1),(X_2)(X_3)) \;\in\; \{ \;& (y_\eta y_{\eta\eta\eta\eta},\, \tfrac{Ch}{\beta'},\, y_{\eta\eta\eta\eta}) \\
& (y_{\eta\eta} y_{\eta\eta\eta},\, \tfrac{Ch}{\beta'^2},\, y_\eta y_{\eta\eta}) \\
& (y_{\eta\eta\eta\eta\eta\eta},\, \tfrac{Ch}{\beta'^3},\, y_\eta y_{\eta\eta}) \;\}\,.
\end{aligned}
$$

Hence if we take C and h sufficiently small, in equations (24),(25) and (28),(29) the associated perturbative terms (X_1) are all strongly dominated by a (X_3) term from the target KdV or MKdV equations.

B. Controllers Derived from the MKdV equation

Applying a similar treatment to (12) and (17), the discretization approximations write respectively

$$
\begin{aligned}
y_{tt} =& \,\omega(h^2 y_{xx} + \tfrac{h^4}{12} y_{xxxx} + \tfrac{h^6}{360} y_{xxxxxx} + O(h^8)) \\
& -(\gamma+\omega)(\tfrac{h^4}{12} y_{xxxx} + \tfrac{h^6}{360} y_{xxxxxx} + O(h^8)) \\
& -\beta'(3h^4 y_x^2 y_{xx} + \tfrac{h^6}{8} y^3 xx + \tfrac{h^7}{4} y_{xx}^2 y_{xxx} + O(h^8))
\end{aligned} \tag{26}
$$

$$
\begin{aligned}
y_{tt} =& \,\gamma(h^2 y_{xx} + \tfrac{h^4}{12} y_{xxxx} + \tfrac{h^6}{360} y_{xxxxxx} + O(h^8)) \\
& +\beta'(3h^4 y_x^2 y_{xx} + \tfrac{h^6}{8} y^3 xx + \tfrac{h^7}{4} y_{xx}^2 y_{xxx} + O(h^8))
\end{aligned} \tag{27}
$$

and working back the change of variables yields

$$
\begin{aligned}
0 \;=\;& y_{\tau\eta} + 36\beta' y_\eta^2 y_{\eta\eta} + \gamma y_{\eta\eta\eta\eta} \\
& + 6h^2\beta' y_{\eta\eta}^3 + 18h\beta' y_\eta y_{\eta\eta}^2 \\
& + \tfrac{h^2}{30}\gamma y_{\eta\eta\eta\eta\eta\eta} + \tfrac{h^2}{48\omega} y_{\tau\tau} + O(h^3)\,;
\end{aligned} \tag{28}
$$

$$
\begin{aligned}
0 \;=\;& y_{\tau\eta} + 36\beta' y_\eta^2 y_{\eta\eta} + \gamma y_{\eta\eta\eta\eta} \\
& + 6h^2\beta' y_{\eta\eta}^3 + 18h\beta' y_\eta y_{\eta\eta}^2 \\
& + \tfrac{h^2}{30}\gamma y_{\eta\eta\eta\eta\eta\eta} - \tfrac{h^2}{48\gamma} y_{\tau\tau} + O(h^3)\,.
\end{aligned} \tag{29}
$$

The first line of (28) and (29) represents the MKdV equation with $v = y_\eta$. Again, plugging in the soliton solution (5), we see that the term in $y_{\tau\tau}$ is negligible with respect to the one in $y_{\tau\eta}$ provided Ch^2 is small. We can further establish that the order of magnitude of (X_1) is (X_2) times the magnitude of (X_3), with respectively

$$
\begin{aligned}
((X_1),(X_2)(X_3)) \;\in\; \{ \;& (y_{\eta\eta}^3,\, \tfrac{C}{\sqrt{\beta'\gamma}},\, y_{\eta\eta\eta\eta}) \\
& (y_\eta y_{\eta\eta}^2,\, \tfrac{C}{\sqrt{\beta'\gamma}},\, y_\eta y_{\eta\eta}) \\
& (y_{\eta\eta\eta\eta\eta\eta},\, \tfrac{C}{\beta'\gamma},\, y_\eta y_{\eta\eta}) \;\}\,.
\end{aligned}
$$

Thus the approximation is good provided we take C and h sufficiently small. In practice, for large β' these conditions are easier to satisfy, but the well-approximated solitons are more constrained. This is consistent with the fact that once the conditions for a good approximation have been fulfilled for a given β', we know by rescaling that *exactly* the same behavior will hold in the vehicle chain, but just at a different scale, when we change β'.

In order to rigorously justify that we can neglect such dominated terms, in particular the higher-order derivative, an elaborate theory of singular perturbations on PDEs would have to be used. This is beyond the scope of the present paper. It is current practice in PDE discretization that such terms can be safely neglected for stable schemes. Involving a higher number of neighboring vehicles in the control law would allow better approximations in this sense; in the PDE discretization this would be called a scheme of higher order.

Arash Farnam received the B.Sc. degree in electrical engineering from the University of Tabriz, Iran, in 2010 and the M.Sc. degree in Control Systems Engineering from the Sahand University of Technology, Iran, in 2012. He is currently working toward the Ph.D. degree at the Department of Electronics and Information Systems (ELIS), Ghent University, Belgium. His research interests include Distributed/Networked control systems, transportation systems, multi-agent systems and time-delay systems.

Alain Sarlette received the engineering degree in applied physics and the Ph.D. degree in applied sciences from the University of Liege, Liege, Belgium, in 2005 and 2009, respectively.

He has held visiting positions at Princeton University (2006) and Ecole des Mines de Paris(2009?2010). Since 2011, he has been a Lecturer at Ghent University, Ghent, Belgium. Since 2014 he is Senior Researcher at INRIA, Paris. His research interests include nonlinear systems and geometric control approaches, coordination algorithms and distributed control, and quantum control.