Well-posedness and convergence of the Lindblad master equation for a quantum harmonic oscillator with multi-photon drive and damping *

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Abstract

We prove that a quantum harmonic oscillator governed by a Lindblad master equation where the typical drive and loss channels are multi-photon processes instead of single-photon ones, converges to a protected subspace spanned by a finite set of coherent states with the same amplitude and uniformly distributed phases. We also show that this convergence features a finite set of bounded invariant observables (Hermitian operators), such that the final state in the protected subspace can be directly predicted from the initial state. The proof includes the full arguments towards the well-posedness of the corresponding dynamics in proper Banach spaces of Hermitian trace-class operators equipped with adapted nuclear norms. It relies on Hille-Yosida theorem and Lyapunov convergence analysis.

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1 Introduction: open quantum systems

The state of an isolated quantum system is notably described by a wave-function $|\psi\rangle$ on a separable Hilbert space $\mathcal{H}$. The evolution of $|\psi(t)\rangle$ is described by the Schrödinger equation

$$\tfrac{d}{dt}|\psi\rangle = \frac{-i}{\hbar}\, H\, |\psi\rangle$$

where the Hamiltonian H is a Hermitian operator on $\mathcal{H}$. This equation implies a unitary evolution in $\mathcal{H}$, i.e. denoting $\langle\psi|\phi\rangle$ the scalar product between $\psi, \phi \in \mathcal{H}$ and $\| |\psi\rangle \|_{\mathcal{H}} = \sqrt{\langle\psi|\psi\rangle}$ the associated norm, we start with a normalized wave-function $\| |\psi(0)\rangle \|_{\mathcal{H}} = 1$ and we are ensured to keep $\| |\psi(t)\rangle \|_{\mathcal{H}} = 1$ for all t. Thus under such so-called Hamiltonian evolution, the Hilbert-Schmidt distance $\| |\psi(t)\rangle - |\phi(t)\rangle \|_{\mathcal{H}}$ between two different initial states remains invariant in time. This is not suitable for control purposes, where we want to drive an initially unknown state towards a target value.

Doing the latter thus requires to consider open quantum systems, i.e. systems interacting with their environment (see [8, Chapter 4] for a recent physical introduction to decoherence and [3, 6] for more formal and mathematical presentations). The most drastic interaction of this type is the famous projective measurement described by Von Neumann, described by an Hermitian operator $Q : \mathcal{H} \mapsto \mathcal{H}$ with spectral decomposition and where the state gets projected onto the eigenspace of Q corresponding to the measurement result. At the other end of the interaction spectrum, the target quantum system can be in weak interaction with an unobserved large environment which rapidly forgets its past state. We can then only describe the expected evolution of the target system, which under appropriate assumptions ensuring essentially the Markovian character of the evolution, follows a so-called Lindblad master equation [3],[14, Chapter 8.4],[8, Chapter 4]

$$\tfrac{d}{dt}\rho = \tfrac{-i}{\hbar}\,[\boldsymbol{H}, \rho] + \sum_j \boldsymbol{L}_j \rho \boldsymbol{L}_j^\dagger - \tfrac{1}{2}\boldsymbol{L}_j^\dagger \boldsymbol{L}_j \rho - \tfrac{1}{2}\rho \boldsymbol{L}_j^\dagger \boldsymbol{L}_j \,.$$

Here the $\boldsymbol{L}_j$ can be a priori arbitrary operators on $\mathcal{H}$, $\boldsymbol{L}_j^\dagger$ is the adjoint (i.e., conjugate Hermitian) of $\boldsymbol{L}_j$ and ρ is a density operator, i.e. a nonnegative Hermitian (i.e., self-adjoint) operator on $\mathcal{H}$ with $\mathrm{Tr}\,(\rho) = 1$. When $\rho = |\psi\rangle\langle\psi|$ i.e. $\mathrm{Rank}\,(\rho) = 1$, and $\boldsymbol{L}_j = 0\ \forall j$, we recover the Schrödinger equation.

Lindblad type evolution, unlike Schrödinger type evolution, can make the state ρ converge asymptotically towards a subspace or a unique state [14, Chapter 8.4],[20, 21]. From a control engineering viewpoint, it is hence tempting to design a Lindblad type system such that it stabilizes some target states thanks to the interaction of the system with an environment, much like the Watt governor does for the steam engine [11]. In quantum control this is called *reservoir engineering* and it has been successfully applied to stabilize quantum states of interest without requiring explicit measurement feedback, see e.g. [16, 15, 17, 13, 7, 18, 9, 10]. Besides the technological advantage of working without sensor feedback, this also enables deterministic stabilization, since the Lindblad equation describes a deterministic evolution whereas individual quantum measurements follow a stochastic process.

Several reservoir engineering proposals, with potential benefits for quantum technology applications, consider infinite-dimensional Hilbert spaces like the ones describing harmonic oscillation of structures (phonons) or of an electromagnetic field at a given frequency [16, 15, 17]. The latter is subject to intense development in cavity- or circuit-Quantum ElectroDynamics experiments and truly quantum states of the electromagnetic field, with no classical equivalent, have been stabilized experimentally. In particular, our collaborators have recently proposed a scheme to stabilize "k-legged Schrödinger cat states", i.e. a quantum superposition of k electromagnetic field states with the same amplitude but k different phases [10]. From physical arguments and an invariance analysis, they argue that their engineered reservoir stabilizes a subspace of all possible electromagnetic field states, spanned by k such states. Such so-called protected subspace or decoherence-free subspace could then be used for encoding quantum information in quantum memories or quantum telecommunication applications [12]. The purpose of the present paper is to rigorously establish the convergence properties of this engineered reservoir in the infinite-dimensional framework.

From a mathematical viewpoint, rigorous analysis of Lindblad type dynamics with $\mathcal{H}$ infinite-dimensional is nontrivial, as even the appropriate space for solutions $\rho(t)$ has to be specified. Physicists usually rely on a practical combination of physical arguments, invariance analysis, and (at best) convergence analysis of a finite-dimensional truncation to convince themselves of the soundness of a proposed Lindblad evolution, before confirming it by experimental implementation. However it is known that this holds traps, as phenomena like loss of probability mass to infinity can appear in some theoretical models. The presence of unbounded operators $\boldsymbol{L}_j$ in the Lindblad equation requires particular care [5, 6], and with the model of [12, 10] we are precisely in this case. Similar questions and mathematical issues relative to well-posedness and long-time behavior of dissipative infinite dimensional quantum systems have been addressed, for example, in [1] where the density operator ρ governed by a Lindblad master equation is replaced by the Wigner pseudo-probability distribution governed by a $3D$ integro-partial differential equations describing the evolution of an electron ensemble connected to an idealized heat bath under the single-particle Hartree approximation.

The contribution of the present paper is precisely to provide a rigorous analysis for the Lindblad dynamics proposed in [12, 10]. We define a well-posed solution space (Section 3, theorem 1); we prove the asymptotic convergence, as desired, of any initial state within this solution space towards a decoherence-free subspace of dimension k, spanned by the k-legged Schrödinger cat states (Section 4, theorem 2); and we characterize the limit point attained by any initial state by establishing the existence of k^2 physical quantities, attached to k^2 linear bounded operators on $\mathcal{H}$, which remain invariant under the Lindblad evolution (Section 4, theorem 3). The essential ingredients for our approach, partially inspired from [5], are:

- the positivity of the density operators ρ and associated trace-class operators, or their decomposition into positive and negative parts;

- a particular commutation property for our specific system, namely that the Lindblad operator $\boldsymbol{L}$ describing such k-photon exchanges is such that $[\boldsymbol{L}, \boldsymbol{L}^\dagger]$ is positive semi-definite;

- building a family of nuclear norms from the Lindblad operator $\boldsymbol{L}$;

- exploiting these norms via the Hille-Yosida theorem for well-posedness, and via the Lyapunov function $\mathrm{Tr}\left(\boldsymbol{L}\rho\boldsymbol{L}^\dagger\right)$ for convergence analysis;

- a density and duality argument to prove the existence of k^2 invariant bounded operators, followed by particular Fourier-transform-like insight to explicitly identify some of them.

Preliminary results, for the case $k = 2$ and without investigating the well-posedness, are available in [2].

2 Lindblad model for harmonic oscillator with k-photon exchange

The harmonic oscillator is the most basic model of a quantum system on an infinite-dimensional Hilbert space [8]. Starting with the canonical orthogonal basis $\{|n\rangle\}_{n\in\mathbb{N}}$ (Fock basis) corresponding to the photon-number states, i.e. $|\psi\rangle = |n\rangle$ corresponds to an oscillator state with exactly n quanta of oscillation, we define the separable Hilbert space

$$\mathcal{H} = \left\{ |\psi\rangle = \sum_{n\in\mathbb{N}} \psi_n |n\rangle \;\middle|\; \psi_n \in \mathbb{C},\ \sum_{n\in\mathbb{N}} |\psi_n|^2 < +\infty \right\} \tag{1}$$

equipped with the usual Hermitian product $\langle\psi|\phi\rangle = \sum_{n\in\mathbb{N}} \psi_n^* \phi_n$ between $|\psi\rangle = \sum_{n\in\mathbb{N}} \psi_n|n\rangle$ and $|\phi\rangle = \sum_{n\in\mathbb{N}} \phi_n|n\rangle$, where a^* denotes the complex conjugate of $a \in \mathbb{C}$.

The set of Hermitian trace-class operators on $\mathcal{H}$ is denoted by $\mathcal{K}^1(\mathcal{H})$. Any $\rho \in \mathcal{K}^1(\mathcal{H})$ is a compact Hermitian operator admitting a spectral decomposition

$$\rho = \sum_{\mu\geq 1} \lambda_\mu |\psi_\mu\rangle\langle\psi_\mu|$$

with $\{|\psi_\mu\rangle\}_{\mu\geq 1}$ a Hilbert basis of $\mathcal{H}$, $\lambda_\mu \in \mathbb{R}$ and $\sum_{\mu\geq 1} |\lambda_\mu| < +\infty$. The state of an open quantum system is described by a *density operator*, which must belong to the set

$$\mathcal{D} = \left\{ \rho \in \mathcal{K}^1(\mathcal{H}) \;\middle|\; \sum_{\mu\geq 1} \lambda_\mu = 1, \;;\; \lambda_\mu \geq 0 \text{ for all } \mu \geq 1 \right\}$$

of positive semidefinite operators with trace one. An interpretation is that λ_μ gives the probability of the quantum system to be in the corresponding state $|\psi_\mu\rangle$, and more generally

$\langle\psi|\rho|\psi\rangle$ is the probability that the system behaves as if it was in the state $|\psi\rangle$, for any normalized $|\psi\rangle$ on $\mathcal{H}$.

The closest quantum equivalent to a classical harmonic oscillator state of amplitude $\alpha \in \mathbb{C}$ is the so-called coherent state

$$|\alpha\rangle = e^{-\frac{|\alpha|^2}{2}} \sum_{n=0}^{\infty} \frac{\alpha^n}{\sqrt{n!}} |n\rangle \,.$$

It is characterized by its invariance under the photon annihilation operator $\boldsymbol{a}$, which is defined by $\boldsymbol{a}|n\rangle = \sqrt{n}|n-1\rangle$ for $n > 0$ and $\boldsymbol{a}|0\rangle = 0$. One indeed checks that $\boldsymbol{a}|\alpha\rangle = \alpha|\alpha\rangle$. For any given integer $k \geq 1$, the aim of the reservoir engineered in [12, 10] is to stabilize a manifold of so-called "k-legged Schrödinger cat states". Such a state is in fact a linear superposition of coherent states, of the form

$$|\psi\rangle = \frac{1}{\vartheta} \sum_{m=1}^{k} e^{i\theta_m} \, |\alpha_m\rangle \,, \qquad \alpha_m = \alpha \, e^{2i\pi m/k} \tag{2}$$

where the θ_m are fixed quantum phases; α can be taken real positive by an appropriate choice of reference frame; and ϑ is a normalization constant that depends on α and $\{\theta_1, ..., \theta_k\}$.

The reservoir proposed in [12, 10] to stabilize such states of the form (2), can be modeled by the following Lindblad master equation for the evolution of the quantum harmonic oscillator's state:

$$\tfrac{d}{dt}\rho = \mathfrak{L}(\rho) = \boldsymbol{L}\rho\boldsymbol{L}^\dagger - \frac{1}{2}\boldsymbol{L}^\dagger\boldsymbol{L}\rho - \frac{1}{2}\rho\boldsymbol{L}^\dagger\boldsymbol{L} \quad \text{with} \quad \boldsymbol{L} = \boldsymbol{a}^k - \alpha^k \boldsymbol{I} \,. \tag{3}$$

Note that the operator $\boldsymbol{L}$ is unbounded, for any $\alpha \in \mathbb{R}$ and any $k \geq 1$. In this paper we prove the well-posedness and convergence properties of this equation (3), in the infinite-dimensional setting associated to the quantum harmonic oscillator Hilbert space $\mathcal{H}$ as defined in (1).

3 Well-posedness

In the sequel, for any $\rho = \sum_{\mu\geq 1} \lambda_\mu |\psi_\mu\rangle\langle\psi_\mu| \in \mathcal{K}^1(\mathcal{H})$ we use the notation

$$\rho^+ = \sum_{\mu\geq 1} \max(0, \lambda_\mu)|\psi_\mu\rangle\langle\psi_\mu|, \quad \rho^- = \sum_{\mu\geq 1} \max(0, -\lambda_\mu)|\psi_\mu\rangle\langle\psi_\mu| \,.$$

Thus $\rho = \rho^+ - \rho^-$ and $|\rho| = \rho^+ + \rho^-$. Equipped with the trace-norm

$$\|\rho\|_{tr} = \mathrm{Tr}\left(|\rho|\right) = \sum_{\mu=1}^{\infty} |\lambda_\mu| \,,$$

$\mathcal{K}^1(\mathcal{H})$ is a Banach space. For any $\rho \in \mathcal{K}^1(\mathcal{H})$ and any bounded operator $\boldsymbol{B}$ on $\mathcal{H}$, the operators $B\rho$ and ρB are trace-class operators and

$$\mathrm{Tr}\,(B\rho) = \mathrm{Tr}\,(\rho B)\,, \quad \mathrm{Tr}\,(B\rho) \leq \mathrm{Tr}\,(|B\rho|) = \|B\rho\|_{tr} \leq \|B\|\mathrm{Tr}\,(|\rho|) = \|B\|\;\|\rho\|_{tr} \tag{4}$$

with the standard induced operator norm

$$\|B\| = \max\{\|B|\psi\rangle\|_{\mathcal{H}} \;\Big|\; |\psi\rangle \in \mathcal{H},\; \||\psi\rangle\|_{\mathcal{H}} = 1\}\,.$$

We denote by $\mathcal{H}^f$ the sub-space of $\mathcal{H}$ associated to $|\psi\rangle$ involving a finite number of photons:

$$\mathcal{H}^f = \left\{ |\psi\rangle = \sum_{n\in\mathbb{N}} \psi_n |n\rangle \;\Big|\; \psi_n \in \mathbb{C},\; \exists \bar{n} \text{ such that } \psi_n = 0 \text{ for } n > \bar{n} \right\}.$$

$\mathcal{H}^f$ is dense in $\mathcal{H}$. We denote also by $\mathcal{K}^f(\mathcal{H})$, the subspace of $\mathcal{K}^1(\mathcal{H})$ of operators whose range is included in a vector space spanned by a finite number of photon-number states:

$$\mathcal{K}^f(\mathcal{H}) = \left\{ \sum_{n=1}^{\bar{n}} \sum_{n'=1}^{\bar{n}} f_{n,n'} |n\rangle\langle n'| \Big| f_{n,n'} \in \mathbb{C},\; \bar{n} \in \mathbb{N}_{>0} \right\}.$$

$\mathcal{K}^f(\mathcal{H})$ is dense in the Banach space $\mathcal{K}^1(\mathcal{H})$. More details about these operator spaces can be found in [19].

The photon-number operator is defined by $\boldsymbol{N} = \sum_{n\in\mathbb{N}} n\,|n\rangle\langle n|$ and it satisfies $\boldsymbol{a}^\dagger \boldsymbol{a} = \boldsymbol{N}$, whereas $\boldsymbol{a}\boldsymbol{a}^\dagger = \boldsymbol{N} + \boldsymbol{I} = \sum_{n\in\mathbb{N}} (n+1)\,|n\rangle\langle n|$. For any $\nu \in \mathbb{N}$, we denote $(\boldsymbol{N} - \nu\boldsymbol{I})^+$ the Hermitian operator defined by $(\boldsymbol{N} - \nu\boldsymbol{I})^+|n\rangle = (n-\nu)|n\rangle$, for $n > \nu$ and $(\boldsymbol{N} - \nu\boldsymbol{I})^+|n\rangle = 0$ for $n \in \{0, \ldots, \nu\}$. Then simple computations show that

$$[\boldsymbol{L}, \boldsymbol{L}^\dagger] \equiv \boldsymbol{L}\boldsymbol{L}^\dagger - \boldsymbol{L}^\dagger\boldsymbol{L} = \boldsymbol{a}^k(\boldsymbol{a}^\dagger)^k - (\boldsymbol{a}^\dagger)^k\boldsymbol{a}^k = \boldsymbol{M} \tag{5}$$

where $\boldsymbol{M} = (\boldsymbol{N} + \boldsymbol{I})(\boldsymbol{N} + 2\boldsymbol{I})\ldots(\boldsymbol{N} + k\boldsymbol{I}) \;-\; \boldsymbol{N}(\boldsymbol{N} - \boldsymbol{I})^+\ldots(\boldsymbol{N} - (k-1)\boldsymbol{I})^+$ is a positive Hermitian operator, unbounded but diagonal in the Fock basis $\{|n\rangle\}_{n\in\mathbb{N}}$.

Lemma 1. *The operator $\boldsymbol{L}^\dagger\boldsymbol{L}$ admits a spectral decomposition $\boldsymbol{L}^\dagger\boldsymbol{L} = \sum_{\mu=1}^{\infty} d_\mu |g_\mu\rangle\langle g_\mu|$ where $(|g_\mu\rangle)_{\mu\geq 0}$ is an Hilbert basis of $\mathcal{H}$ and $d_\mu \geq 0$.*

Proof. We construct the spectral decomposition of $\boldsymbol{L}^\dagger\boldsymbol{L}$ from the one of the inverse of $\boldsymbol{I} + \lambda\boldsymbol{L}^\dagger\boldsymbol{L}$ for $\lambda > 0$ small enough. Noting that $\boldsymbol{L}^\dagger\boldsymbol{L} = \boldsymbol{N}(\boldsymbol{N} - \boldsymbol{I})^+\ldots(\boldsymbol{N} - k + 1)^+ + \alpha^{2k} - \alpha^k((\boldsymbol{a}^\dagger)^k + \boldsymbol{a}^k)$, we can write

$$\boldsymbol{I} + \lambda\boldsymbol{L}^\dagger\boldsymbol{L} = \Big(\boldsymbol{I} - \lambda\alpha^k\Big((\boldsymbol{a}^\dagger)^k + \boldsymbol{a}^k\Big)\boldsymbol{R}^{-1}\Big)\boldsymbol{R} \tag{6}$$

with $\boldsymbol{R} = \boldsymbol{I} + \lambda\Big(\boldsymbol{N}(\boldsymbol{N} - \boldsymbol{I})^+\ldots(\boldsymbol{N} - k + 1)^+ + \alpha^{2k}\Big)$ being Hermitian and positive. Indeed, despite $\boldsymbol{R}$ being unbounded, it is by definition diagonalizable in the Fock basis $\{|n\rangle\}_{n\in\mathbb{N}}$

and from this diagonal form it has a well-defined inverse $\boldsymbol{R}^{-1}$, whose spectrum is bounded and decays to zero, and thus $\boldsymbol{R}^{-1}$ is compact. We now work towards inverting the product on the right side of (6).

Take $|\psi\rangle = \sum_{n\in\mathbb{N}} \psi_n |n\rangle$ in $\mathcal{H}^f$. Then

$$((\boldsymbol{a}^\dagger)^k + \boldsymbol{a}^k)|\psi\rangle = \sum_{n\geq 0} \sqrt{(n+1)\dots(n+k)}\psi_n|n+k\rangle + \sum_{n\geq k} \sqrt{n(n-1)\dots(n-k+1)}\psi_n|n-k\rangle$$

and a few computations lead to the bound

$$\langle\psi|((\boldsymbol{a}^\dagger)^k + \boldsymbol{a}^k)^2|\psi\rangle \leq 2\sum_{n\geq 0}((n+k)^k + n^k)|\psi_n|^2. \tag{7}$$

Clearly $|\phi\rangle = \boldsymbol{R}^{-1}|\psi\rangle = \sum_{n\geq 0}\phi_n|n\rangle$ belongs also to $\mathcal{H}^f$ since $\phi_n = \frac{\psi_n}{1+\lambda(\max(0,n(n-1)\dots(n-k+1))+\alpha^{2k})}$. Inserting this into (7) gives for any $|\psi\rangle \in \mathcal{H}^f$,

$$\lambda^2\alpha^{2k}\left\langle\psi\left|\boldsymbol{R}^{-1}\Big((\boldsymbol{a}^\dagger)^k + \boldsymbol{a}^k\Big)^2\boldsymbol{R}^{-1}\right|\psi\right\rangle \leq \sum_{n\geq 0}\frac{2\lambda^2\alpha^{2k}((n+k)^k + n^k)}{(1+\lambda(\max(0,n(n\text{-}1)\dots(n\text{-}k+1)) + \alpha^{2k}))^2}|\psi_n|^2.$$

A rough estimation shows that there exists a constant $c > 0$ such that $\forall n \in \mathbb{N}$, we have

$$\frac{2\lambda^2\alpha^{2k}((n+k)^k + n^k)}{(1+\lambda(\max(0,n(n-1)\dots(n-k+1)) + \alpha^{2k}))^2} \leq c\lambda.$$

By density of $\mathcal{H}^f$ into $\mathcal{H}$, we have

$$\forall|\psi\rangle \in \mathcal{H}, \quad \lambda^2\alpha^{2k}\left\langle\psi\left|\boldsymbol{R}^{-1}\Big((\boldsymbol{a}^\dagger)^k + \boldsymbol{a}^k\Big)^2\boldsymbol{R}^{-1}\right|\psi\right\rangle \leq c\lambda\langle\psi|\psi\rangle.$$

thus $\lambda\alpha^k\Big((\boldsymbol{a}^\dagger)^k + \boldsymbol{a}^k\Big)\boldsymbol{R}^{-1}$ is a bounded operator. Moreover for $\lambda < 1/c$ its norm is strictly less than one. This implies that $\Big(\boldsymbol{I} - \lambda\alpha^k\Big((\boldsymbol{a}^\dagger)^k + \boldsymbol{a}^k\Big)\boldsymbol{R}^{-1}\Big)$ is bounded and admits a bounded inverse. We can then define

$$(\boldsymbol{I} + \lambda\boldsymbol{L}^\dagger\boldsymbol{L})^{-1} = \boldsymbol{R}^{-1}\Big(\boldsymbol{I} - \lambda\alpha^k\Big((\boldsymbol{a}^\dagger)^k + \boldsymbol{a}^k\Big)\boldsymbol{R}^{-1}\Big)^{-1}$$

which is the product of a bounded operator with a compact operator, hence it is compact. Thus $(\boldsymbol{I} + \lambda\boldsymbol{L}^\dagger\boldsymbol{L})^{-1}$ is a compact Hermitian operator: it admits a spectral decomposition. By a $\mathcal{H}^f$ density argument, we have $0 \leq \langle\psi|(\boldsymbol{I} + \lambda\boldsymbol{L}^\dagger\boldsymbol{L})^{-1}|\psi\rangle \leq \langle\psi|\psi\rangle$ for all $|\psi\rangle \in \mathcal{H}$. Taking all this together, we can then write

$$(\boldsymbol{I} + \lambda\boldsymbol{L}^\dagger\boldsymbol{L})^{-1} = \sum_{\mu\geq 1} s_\mu|g_\mu\rangle\langle g_\mu|$$

with $0 \leq s_\mu \leq 1$ and $\{|g_\mu\rangle\}_{\mu\geq 1}$ a Hilbert basis. We conclude with $d_\mu = (1/s_\mu - 1)/\lambda$. □

Via the spectral decomposition of lemma 1, $\boldsymbol{S}_\nu = (\boldsymbol{I} + \boldsymbol{L}^\dagger \boldsymbol{L})^\nu$ is well defined for any $\nu \geq 0$ by

$$\boldsymbol{S}_\nu = (\boldsymbol{I} + \boldsymbol{L}^\dagger \boldsymbol{L})^\nu = \sum_{\mu=1}^{\infty} (1 + d_\mu)^\nu |g_\mu\rangle\langle g_\mu| \,.$$

Denote by

$$\mathcal{K}_{L,\nu}(\mathcal{H}) = \left\{ \rho \in \mathcal{K}^1(\mathcal{H}) \;\middle|\; \mathrm{Tr}\left(|\boldsymbol{S}_\nu \rho \boldsymbol{S}_\nu|\right) < +\infty \right\}.$$

We have the following lemma.

Lemma 2. *For any $\nu \geq 0$ the subspace $\mathcal{K}_{L,\nu}(\mathcal{H})$ equipped with the norm*

$$\|\rho\|_{L,\nu} = Tr(|\boldsymbol{S}_\nu \rho \boldsymbol{S}_\nu|) \tag{8}$$

is a Banach space. Moreover $\rho \in \mathcal{K}_{L,\nu}(\mathcal{H})$ for $\nu \geq 1/2$ implies $\boldsymbol{L}\rho\boldsymbol{L}^\dagger \in \mathcal{K}^1(\mathcal{H})$.

Proof. The first statement holds since the operation $\rho \mapsto \boldsymbol{S}_\nu \rho \boldsymbol{S}_\nu$ is an isometry mapping the Banach space $\mathcal{K}^1(\mathcal{H})$, equipped with the trace norm, to the space $\mathcal{K}_{L,\nu}(\mathcal{H})$ equipped with the norm $\|\rho\|_{L,\nu}$.

The second statement follows because the operators $\boldsymbol{B}_\nu = \boldsymbol{L}\,\boldsymbol{S}_\nu^{-1}$ and $\boldsymbol{B}_\nu^\dagger$ are bounded, since $\boldsymbol{B}_\nu^\dagger \boldsymbol{B}_\nu = \boldsymbol{S}_\nu^{-1} \boldsymbol{L}^\dagger \boldsymbol{L} \boldsymbol{S}_\nu^{-1} = \frac{\boldsymbol{L}^\dagger \boldsymbol{L}}{(\boldsymbol{I}+\boldsymbol{L}^\dagger \boldsymbol{L})^{2\nu}}$ is bounded, as soon as $2\nu \geq 1$. Then for $\rho \in \mathcal{K}_{L,\nu}(\mathcal{H})$, defining $\phi = \boldsymbol{S}_\nu \rho \boldsymbol{S}_\nu \in \mathcal{K}^1(\mathcal{H})$, we have that $\boldsymbol{L}\rho\boldsymbol{L}^\dagger = \boldsymbol{B}_\nu \phi \boldsymbol{B}_\nu^\dagger$ is the composition of a trace-class operator ϕ with two bounded operators, hence it is trace-class. ☐

The above considerations put us on track towards the following result.

Theorem 1. *For any integer $k > 0$ and real $\alpha > 0$, the initial value problem*

$$\frac{d}{dt}\rho = \mathfrak{L}(\rho) = \boldsymbol{L}\rho\boldsymbol{L}^\dagger - (\boldsymbol{L}^\dagger \boldsymbol{L}\rho + \rho \boldsymbol{L}^\dagger \boldsymbol{L})/2, \quad \rho(0) = \rho_0 \tag{9}$$

with $\boldsymbol{L} = \boldsymbol{a}^k - \alpha^k \boldsymbol{I}$ is well posed in the Banach space $\mathcal{K}_{L,\nu}(\mathcal{H})$ for any $\nu \geq 1/2$. Moreover, for all t positive, $Tr(\rho(t)) = Tr(\rho_0)$ and $\|\rho(t)\|_{L,\nu} \leq \|\rho_0\|_{L,\nu}$.

This means that for any $\rho_0 \in \mathcal{K}_{L,\nu}(\mathcal{H})$ with $\nu \geq 1/2$, there exists a unique C^1 function: $[0, +\infty[\ni t \mapsto \rho(t) \in \mathcal{K}_{L,\nu}(\mathcal{H})$, such that $\rho(t)$ belongs to the domain of the super-operator $\mathfrak{A}$,

$$\rho \mapsto \mathfrak{A}(\rho) = (\boldsymbol{L}^\dagger \boldsymbol{L}\rho + \rho \boldsymbol{L}^\dagger \boldsymbol{L})/2 - \boldsymbol{L}\rho\boldsymbol{L}^\dagger$$

and satisfies (9).

The proof is based on the Hille-Yosida theorem, recalled in the appendix. It consists in proving that the unbounded super-operator $\mathfrak{A}$ on $\mathcal{K}_{L,\nu}(\mathcal{H})$ is m-accretive. Its domain $D(\mathfrak{A})$ is dense since it contains $\mathcal{K}^f(\mathcal{H})$. The essential difficulty is to prove that for any $\lambda > 0$, $\boldsymbol{I} + \lambda\mathfrak{A}$ is a bijection from $D(\mathfrak{A})$ into $\mathcal{K}_{L,\nu}(\mathcal{H})$, with $(\boldsymbol{I} + \lambda\mathfrak{A})^{-1}$ a bounded linear operator on $\mathcal{K}_{L,\nu}(\mathcal{H})$ with norm less or equal to 1. The proof is partially inspired by [5]: it is decomposed into the successive lemmas 3 to 5. One of the key and original arguments is that the commutator $[\boldsymbol{L}, \boldsymbol{L}^\dagger] = \boldsymbol{M}$ defines a non-negative Hermitian (unbounded) operator, see (5).

Lemma 3. *For any* $f \in \mathcal{K}_{L,\nu}(\mathcal{H})$ *with* $\nu \geq 0$*, any* $\lambda > 0$ *and* $r \in [0,1)$*, there exists a unique* $\rho_r \in \mathcal{K}_{L,\nu}(\mathcal{H})$ *solution of*

$$\tfrac{\boldsymbol{I}+\lambda \boldsymbol{L}^\dagger \boldsymbol{L}}{2}\rho_r + \rho_r \tfrac{\boldsymbol{I}+\lambda \boldsymbol{L}^\dagger \boldsymbol{L}}{2} = f + r\lambda \boldsymbol{L}\rho_r \boldsymbol{L}^\dagger \,. \tag{10}$$

Moreover we have $\|\rho_r\|_{L,\nu} \leq \|f\|_{L,\nu}$*. When additionally* $f \geq 0$*, we have* $0 \leq \rho_{r_1} \leq \rho_{r_2}$ *for* $0 \leq r_1 \leq r_2 < 1$ *;*

Proof. It is proved in Lemma 1, that $(I + \lambda \boldsymbol{L}^\dagger \boldsymbol{L})^{-1}$ is a compact strictly positive operator on $\mathcal{H}$ and $(I + \lambda \boldsymbol{L}^\dagger \boldsymbol{L})^{-1} \leq \boldsymbol{I}$. Thus it generates a strongly continuous semigroup $\{e^{-s(\boldsymbol{I}+\lambda \boldsymbol{L}^\dagger \boldsymbol{L})}\}_{s\geq 0}$ of contractions on $\mathcal{H}$. Hence for any $\xi \in \mathcal{K}_{L,\nu}(\mathcal{H})$, equivalently for any $\phi = \boldsymbol{S}_\nu \xi \boldsymbol{S}_\nu \in \mathcal{K}^1(\mathcal{H})$, there exists a unique solution $\rho \in \mathcal{K}_{L,\nu}(\mathcal{H})$ of the Sylvester equation

$$\left(\tfrac{\boldsymbol{I}+\lambda \boldsymbol{L}^\dagger \boldsymbol{L}}{2}\right)\rho + \rho\left(\tfrac{\boldsymbol{I}+\lambda \boldsymbol{L}^\dagger \boldsymbol{L}}{2}\right) = \xi = \boldsymbol{S}_\nu^{-1}\phi \boldsymbol{S}_\nu^{-1}$$

and given by the usual formula

$$\begin{aligned} \rho \;=\; \Pi(\xi) &= \int_0^{+\infty} e^{-s(\boldsymbol{I}+\lambda \boldsymbol{L}^\dagger \boldsymbol{L})/2}\,\xi\, e^{-s(\boldsymbol{I}+\lambda \boldsymbol{L}^\dagger \boldsymbol{L})/2}\,ds \\ &= \int_0^{+\infty} \frac{e^{-s(\boldsymbol{I}+\lambda \boldsymbol{L}^\dagger \boldsymbol{L})/2}}{\boldsymbol{S}_\nu}\,\phi\,\frac{e^{-s(\boldsymbol{I}+\lambda \boldsymbol{L}^\dagger \boldsymbol{L})/2}}{\boldsymbol{S}_\nu}\,ds. \end{aligned} \tag{11}$$

Note that the semigroup generator $e^{-s(\boldsymbol{I}+\lambda \boldsymbol{L}^\dagger \boldsymbol{L})/2}$ commutes with $\boldsymbol{S}_\nu$, so in fact $\Pi(\phi) = \Pi(\boldsymbol{S}_\nu \xi \boldsymbol{S}_\nu) = \boldsymbol{S}_\nu \Pi(\phi)\boldsymbol{S}_\nu$. Inserting the expression of Π into (10) gives

$$\xi = f + r\lambda \boldsymbol{L}\Pi(\xi)\boldsymbol{L}^\dagger = f + r\mathfrak{B}(\xi) \quad \text{with } \mathfrak{B}(\xi) = \lambda \boldsymbol{L}\Pi(\xi)\boldsymbol{L}^\dagger \,.$$

We will conclude the existence and uniqueness proof by showing that $\xi \mapsto f + r\mathfrak{B}(\xi)$ is a strict contraction for the $\|\cdot\|_{L,\nu}$-norm on $\mathcal{K}_{L,\nu}(\mathcal{H})$. Each part of the proof first considers the case of positive operators, then (between $\rhd$ $\lhd$) adapts it to arbitrary ones.

- *Step 1:* Contraction, monotonicity of $\Pi(\xi)$ and some trace estimates.

From (11) it is obvious that $\Pi(\xi) \geq 0$ for any $\xi \geq 0$ in $\mathcal{K}_{L,\nu}(\mathcal{H})$, hence by linearity $\Pi(\xi_1) \leq \Pi(\xi_2)$ as soon as $\xi_1 \leq \xi_2$ in $\mathcal{K}_{L,\nu}(\mathcal{H})$, i.e. Π is monotone. Since $\boldsymbol{S}_\nu \boldsymbol{L} e^{-s(\boldsymbol{I}+\lambda \boldsymbol{L}^\dagger \boldsymbol{L})/2}\boldsymbol{S}_\nu^{-1}$ is a bounded operator for any $\nu \geq 0$ and any $s > 0$, we can write thanks to (4), for any $\boldsymbol{S}_\nu^{-1}\phi \boldsymbol{S}_\nu^{-1} = \xi \geq 0$ in $\mathcal{K}_{L,\nu}(\mathcal{H})$:

$$\left\|\boldsymbol{L}\Pi(\xi)\boldsymbol{L}^\dagger\right\|_{L,\nu} = \mathrm{Tr}\left(\boldsymbol{S}_\nu \boldsymbol{L}\Pi(\xi)\boldsymbol{L}^\dagger \boldsymbol{S}_\nu\right)$$

$$
\begin{aligned}
&= \mathrm{Tr}\left(\int_0^{+\infty} \boldsymbol{S}_\nu^{-1}\, e^{-s(\boldsymbol{I}+\lambda \boldsymbol{L}^\dagger \boldsymbol{L})/2} \boldsymbol{L}^\dagger \boldsymbol{S}_\nu^2 \boldsymbol{L} e^{-s(\boldsymbol{I}+\lambda \boldsymbol{L}^\dagger \boldsymbol{L})/2} \boldsymbol{S}_\nu^{-1}\ \phi\, ds\right) \\
&= \mathrm{Tr}\left(\int_0^{+\infty} \boldsymbol{S}_\nu^{-1}\, e^{-s\cdots} \boldsymbol{L}^\dagger (\boldsymbol{I}+\boldsymbol{L}\boldsymbol{L}^\dagger - \boldsymbol{M})^{2\nu} \boldsymbol{L} e^{-s\cdots} \boldsymbol{S}_\nu^{-1}\ \phi\, ds\right) \\
&\leq \mathrm{Tr}\left(\int_0^{+\infty} \boldsymbol{S}_\nu^{-1}\, e^{-s\cdots} \boldsymbol{L}^\dagger (\boldsymbol{I}+\boldsymbol{L}\boldsymbol{L}^\dagger)^{2\nu} \boldsymbol{L} e^{-s\cdots} \boldsymbol{S}_\nu^{-1}\ \phi\, ds\right) \\
&= \mathrm{Tr}\left(\int_0^{+\infty} \boldsymbol{S}_\nu^{-1}\, e^{-s\cdots} (\boldsymbol{L}^\dagger \boldsymbol{L}) (\boldsymbol{I}+\boldsymbol{L}^\dagger\boldsymbol{L})^{2\nu} e^{-s\cdots} \boldsymbol{S}_\nu^{-1}\ \phi\, ds\right) \\
&= \mathrm{Tr}\left(\int_0^{+\infty} e^{-s(\boldsymbol{I}+\lambda \boldsymbol{L}^\dagger \boldsymbol{L})}\, ds\ \boldsymbol{L}^\dagger \boldsymbol{L} \phi\right) \\
&= \mathrm{Tr}\left(\frac{\boldsymbol{L}^\dagger \boldsymbol{L}}{\boldsymbol{I}+\lambda \boldsymbol{L}^\dagger \boldsymbol{L}} \phi\right) \leq \tfrac{1}{\lambda} \mathrm{Tr}\,(\phi) = \tfrac{1}{\lambda} \|\xi\|_{L,\nu}\ . \qquad (12)
\end{aligned}
$$

Here we have used that $\int_0^{+\infty} e^{-s\left(\boldsymbol{I}+\lambda \boldsymbol{L}^\dagger \boldsymbol{L}\right)} ds = \frac{\boldsymbol{I}}{\boldsymbol{I}+\lambda \boldsymbol{L}^\dagger \boldsymbol{L}}$, the commutation relation with $\boldsymbol{M}$ from (5) and the identity $(\boldsymbol{I}+\boldsymbol{L}\boldsymbol{L}^\dagger)^{2\nu} \boldsymbol{L} = \boldsymbol{L}(\boldsymbol{I}+\boldsymbol{L}^\dagger \boldsymbol{L})^{2\nu}$. We can repeat a similar argument with the bounded operator $e^{-s(\boldsymbol{I}+\lambda \boldsymbol{L}^\dagger \boldsymbol{L})/2}$ to get

$$
\|\Pi(\xi)\|_{L,\nu} = \mathrm{Tr}\,(\Pi(\phi)) = \mathrm{Tr}\left(\frac{\boldsymbol{I}}{\boldsymbol{I}+\lambda \boldsymbol{L}^\dagger \boldsymbol{L}} \phi\right) \leq \mathrm{Tr}\,(\phi) = \|\xi\|_{L,\nu}\ . \qquad (13)
$$

$\triangleright$ For $\xi \not\geq 0$ in $\mathcal{K}_{L,\nu}(\mathcal{H})$, writing $\phi = \phi^+ - \phi^-$ we have $\phi^+, \phi^- \in \mathcal{K}^1(\mathcal{H})$ nonnegative and such that $|\phi| = |\boldsymbol{S}_\nu \xi \boldsymbol{S}_\nu| = \phi^+ + \phi^-$. Then we get, using the above, linearity and monotonicity of Π, and the triangular inequality for the trace-norm:

$$
\begin{aligned}
\|\Pi(\xi)\|_{L,\nu} &= \mathrm{Tr}\left(|\Pi(\phi^+) - \Pi(\phi^-)|\right) \leq \mathrm{Tr}\left(\Pi(\phi^+)\right) + \mathrm{Tr}\left(\Pi(\phi^-)\right) \\
&\leq \mathrm{Tr}\left(\phi^+\right) + \mathrm{Tr}\left(\phi^-\right) = \|\xi\|_{L,\nu}\,. \qquad \triangleleft
\end{aligned}
$$

Thus Π is a (non-strict) contraction for the $\|\cdot\|_{L,\nu}$-norm on $\mathcal{K}_{L,\nu}(\mathcal{H})$.

• *Step 2:* Contraction of $\mathfrak{B}(\xi)$ and consequences.

Let us prove that $\xi \mapsto \mathfrak{B}(\xi) = \lambda \boldsymbol{L} \Pi(\xi) \boldsymbol{L}^\dagger$ is a (non-strict) contraction for the $\|\cdot\|_{L,\nu}$-norm on $\mathcal{K}_{L,\nu}(\mathcal{H})$. For any $\xi \in \mathcal{K}_{L,\nu}(\mathcal{H})$ nonnegative, ϕ and $\mathfrak{B}(\xi)$ are nonnegative. We deduce from (12),(13) that $\mathfrak{B}(\xi)$ belongs to $\mathcal{K}_{L,\nu}(\mathcal{H})$ with

$$
\|\mathfrak{B}(\xi)\|_{L,\nu} \leq \lambda \mathrm{Tr}\left(\frac{\boldsymbol{L}^\dagger \boldsymbol{L}}{\boldsymbol{I}+\lambda \boldsymbol{L}^\dagger \boldsymbol{L}} \phi\right) = \mathrm{Tr}\,(\phi) - \mathrm{Tr}\,(\Pi(\phi)) \ \leq\ \mathrm{Tr}\,(\phi)\ . \qquad (14)
$$

▷ For $\xi \in \mathcal{K}_{L,\nu}(\mathcal{H})$ arbitrary, decompose $\phi \ngeq 0$ into $\phi = \phi^+ - \phi^- \in \mathcal{K}^1(\mathcal{H})$ and similarly to above we get:

$$\begin{aligned}
\|\mathfrak{B}(\xi)\|_{L,\nu} &= \mathrm{Tr}\left(\left|\boldsymbol{S}_\nu \mathfrak{B}(\boldsymbol{S}_\nu^{-1}\phi^+\boldsymbol{S}_\nu^{-1})\boldsymbol{S}_\nu - \boldsymbol{S}_\nu \mathfrak{B}(\boldsymbol{S}_\nu^{-1}\phi^-\boldsymbol{S}_\nu^{-1})\boldsymbol{S}_\nu\right|\right) \\
&\leq \mathrm{Tr}\left(\left|\boldsymbol{S}_\nu \mathfrak{B}(\boldsymbol{S}_\nu^{-1}\phi^+\boldsymbol{S}_\nu^{-1})\boldsymbol{S}_\nu\right|\right) + \mathrm{Tr}\left(\left|\boldsymbol{S}_\nu \mathfrak{B}(\boldsymbol{S}_\nu^{-1}\phi^-\boldsymbol{S}_\nu^{-1})\boldsymbol{S}_\nu\right|\right) \\
&\leq \mathrm{Tr}\left(\phi^+\right) + \mathrm{Tr}\left(\phi^-\right) \;=\; \mathrm{Tr}\left(|\phi|\right) \;=\; \|\xi\|_{L,\nu} \ ,
\end{aligned}$$

since on the second line we can drop the absolute value. ◁

This proves (non-strict) contraction of $\mathfrak{B}(\xi)$. Thus $\xi \mapsto f + r\mathfrak{B}(\xi)$ is a strict contraction on $\mathcal{K}_{L,\nu}(\mathcal{H})$ as soon as $r \in [0,1)$. Consequently, it admits a unique fixed point $\xi_r \in \mathcal{K}_{L,\nu}(\mathcal{H})$ given by the absolutely converging series

$$\xi_r = \mathfrak{C}_r(f) = \sum_{s=0}^{+\infty} r^s \mathfrak{B}^s(f). \tag{15}$$

This justifies a posteriori that taking $\xi \in \mathcal{K}_{L,\nu}(\mathcal{H})$ for the Sylvester equation yields a valid result. The solution ρ_r is then given by $\rho_r = \Pi(\mathfrak{C}_r(f))$.

• *Step 3:* There remains to prove the inequalities.

First take $f \geq 0$ and $0 \leq r_1 \leq r_2 < 1$. Then by (15) we have $0 \leq f \leq \mathfrak{C}_{r_1}(f) \leq \mathfrak{C}_{r_2}(f)$, in particular $\xi_r \geq 0$ for all $r \in [0,1)$. Since Π is monotone, we have $0 \leq \Pi(f) \leq \rho_{r_1} \leq \rho_{r_2}$ which proves the first inequality. Moreover, we have

$$\begin{aligned}
\mathrm{Tr}\left(\boldsymbol{S}_\nu \rho_r \boldsymbol{S}_\nu\right) &= \mathrm{Tr}\left(\boldsymbol{S}_\nu \Pi(\xi_r)\boldsymbol{S}_\nu\right) = \mathrm{Tr}\left(\Pi(\phi_r)\right) \leq \mathrm{Tr}\left(\phi_r\right) - \mathrm{Tr}\left(\boldsymbol{S}_\nu \mathfrak{B}(\xi_r)\boldsymbol{S}_\nu\right) \\
&= \mathrm{Tr}\left(\boldsymbol{S}_\nu f \boldsymbol{S}_\nu\right) + (r-1)\mathrm{Tr}\left(\boldsymbol{S}_\nu \mathfrak{B}(\xi_r)\boldsymbol{S}_\nu\right) \leq \mathrm{Tr}\left(\boldsymbol{S}_\nu f \boldsymbol{S}_\nu\right) \ .
\end{aligned}$$

At the end of the first line we have used (14); the next equality comes from the definition $\phi_r = \boldsymbol{S}_\nu \xi_r \boldsymbol{S}_\nu = \boldsymbol{S}_\nu(f + r\mathfrak{B}(\xi_r))\boldsymbol{S}_\nu$ where all terms have been proved to be trace-class, and the final inequality holds because $f \geq 0$ implies $\xi_r \geq 0$ and thus $\mathfrak{B}(\xi_r) \geq 0$. This would conclude the proof if we impose $f \geq 0$.

▷ To prove that $\mathrm{Tr}\left(|\boldsymbol{S}_\nu \rho_r \boldsymbol{S}_\nu|\right) \leq \mathrm{Tr}\left(|\boldsymbol{S}_\nu f \boldsymbol{S}_\nu|\right)$ for $f \ngeq 0$ in $\mathcal{K}_{L,\nu}(\mathcal{H})$, define $g = \boldsymbol{S}_\nu f \boldsymbol{S}_\nu \in \mathcal{K}^1(\mathcal{H})$ and decompose $g = g^+ - g^-$ with $g^+, g^- \geq 0$. Then define $f^p = \boldsymbol{S}_\nu^{-1} g^+ \boldsymbol{S}_\nu^{-1}$ and the associated solution ρ_r^p of (10), and similarly for f^n and ρ_r^n. By construction, $f^p, f^n, \rho_r^p, \rho_r^n$ are nonnegative and belong to $\mathcal{K}_{L,\nu}(\mathcal{H})$. Note however that nothing guarantees that e.g. $\rho_r^p = \rho_r^+$, such that although $\rho_r = \rho_r^p - \rho_r^n$ by linearity, possibly $|\rho_r| = \rho_r^+ + \rho_r^- \neq \rho_r^p + \rho_r^n$. The triangular inequality for the trace-norm nevertheless guarantees

$$\begin{aligned}
\|\rho_r\|_{L,\nu} &= \|\rho_r^p - \rho_r^n\|_{L,\nu} \leq \|\rho_r^p\|_{L,\nu} + \|\rho_r^n\|_{L,\nu} = \mathrm{Tr}\left(\boldsymbol{S}_\nu \rho_r^p \boldsymbol{S}_\nu\right) + \mathrm{Tr}\left(\boldsymbol{S}_\nu \rho_r^n \boldsymbol{S}_\nu\right) \\
&\leq \mathrm{Tr}\left(\boldsymbol{S}_\nu f^p \boldsymbol{S}_\nu\right) + \mathrm{Tr}\left(\boldsymbol{S}_\nu f^n \boldsymbol{S}_\nu\right) = \mathrm{Tr}\left(g^+ + g^-\right) = \mathrm{Tr}\left(|g|\right) = \|f\|_{L,\nu} \ .
\end{aligned}$$

The second inequality is obtained thanks to the property $\mathrm{Tr}\left(\boldsymbol{S}_\nu \rho_r \boldsymbol{S}_\nu\right) \leq \mathrm{Tr}\left(\boldsymbol{S}_\nu f \boldsymbol{S}_\nu\right)$ just proved for $f \geq 0$. The equalities use linearity and positivity. ◁ □

Lemma 4. *Take a nonnegative $f \in \mathcal{K}_{L,\nu}(\mathcal{H})$ with $\nu \geq 1/2$ and consider $\rho_r \in \mathcal{K}_{L,\nu}(\mathcal{H})$ given by Lemma 3. Then, for r tending to 1^-, ρ_r converges towards ρ in $\mathcal{K}_{L,\nu}(\mathcal{H})$ solution of*

$$\rho + \lambda\mathfrak{A}(\rho) = \tfrac{\boldsymbol{I}+\lambda \boldsymbol{L}^\dagger \boldsymbol{L}}{2}\rho + \rho\tfrac{\boldsymbol{I}+\lambda \boldsymbol{L}^\dagger \boldsymbol{L}}{2} - \lambda \boldsymbol{L}\rho \boldsymbol{L}^\dagger = f. \tag{16}$$

Moreover $\rho \geq 0$ and $\|\rho\|_{L,\nu} = Tr(\boldsymbol{S}_\nu \rho \boldsymbol{S}_\nu) \leq Tr(\boldsymbol{S}_\nu f \boldsymbol{S}_\nu) = \|f\|_{L,\nu}$.

Proof. Take an increasing sequence $\{r_k\}_{k\in\mathbb{N}}$ in $[0,1)$ converging to 1. From $\left\|\rho_{r_{k_1}}\right\|_{L,\nu} \leq \left\|\rho_{r_{k_2}}\right\|_{L,\nu} \leq \|f\|_{L,\nu}$ for $k_1 \leq k_2$, as proved in Lemma 3, we deduce that with $s_k = \|\rho_{r_k}\|_{L,\nu} = \mathrm{Tr}\left(\boldsymbol{S}_\nu \rho_{r_k} \boldsymbol{S}_\nu\right)$, the sequence $\{s_k\}_{k\in\mathbb{N}}$ is positive, increasing and bounded by $\|f\|_{L,\nu}$. Thus it converges. Moreover, Lemma 3 proves that $\rho_{r_{k_2}} - \rho_{r_{k_1}} \geq 0$ for $k_1 \leq k_2$, such that

$$\left\|\rho_{r_{k_2}} - \rho_{r_{k_1}}\right\|_{L,\nu} = \mathrm{Tr}\left(\boldsymbol{S}_\nu(\rho_{r_{k_2}} - \rho_{r_{k_1}})\boldsymbol{S}_\nu\right) = s_{k_2} - s_{k_1}$$

and thus $\{\rho_{r_k}\}_{k\in\mathbb{N}}$ is a Cauchy sequence in the Banach space $\mathcal{K}_{L,\nu}(\mathcal{H})$ equipped with the $\|\cdot\|_{L,\nu}$-norm. Thus, it converges to some limit $\rho \in \mathcal{K}_{L,\nu}(\mathcal{H})$. This ρ is independent of the increasing sequence r_k tending to 1, and $\|\rho\|_{L,\nu}$ is the limit of the sequence $\{s_k\}_{k\in\mathbb{N}}$, which is bounded by $\|f\|_{L,\nu}$.

For any $r \in [0,1)$, we have defined ρ_r to be solution of

$$\tfrac{\boldsymbol{I}+\lambda \boldsymbol{L}^\dagger \boldsymbol{L}}{2}\rho_r + \rho_r\tfrac{\boldsymbol{I}+\lambda \boldsymbol{L}^\dagger \boldsymbol{L}}{2} - \lambda \boldsymbol{L}\rho_r \boldsymbol{L}^\dagger = f - (1-r)\lambda \boldsymbol{L}\rho_r \boldsymbol{L}^\dagger.$$

But for $\nu \geq 1/2$, $\boldsymbol{L}\rho_r \boldsymbol{L}^\dagger$ is trace-class (see Lemma 2) and it converges to $\boldsymbol{L}\rho \boldsymbol{L}^\dagger$ in the trace-norm topology. Hence the left-hand side is also trace-class, and it converges to $\tfrac{\boldsymbol{I}+\lambda \boldsymbol{L}^\dagger \boldsymbol{L}}{2}\rho + \rho\tfrac{\boldsymbol{I}+\lambda \boldsymbol{L}^\dagger \boldsymbol{L}}{2} - \lambda \boldsymbol{L}\rho \boldsymbol{L}^\dagger$ in the trace-norm topology. Thus by taking the limit in $\mathcal{K}^1(\mathcal{H})$ of the above equality, we get (16). □

The final Lemma proves the bijection property of $\boldsymbol{I} + \lambda\mathfrak{A}$, with $\|(\boldsymbol{I} + \lambda\mathfrak{A})^{-1}\| \leq 1$, such that we can conclude with the Hille-Yosida (Theorem 4 recalled in appendix).

Lemma 5. *Take $f \in \mathcal{K}_{L,\nu}(\mathcal{H})$ with $\nu \geq 1/2$, and $\lambda > 0$. Then there exists a unique $\rho \in \mathcal{K}_{L,\nu}(\mathcal{H})$ solution of $\rho + \lambda\mathfrak{A}(\rho) = f$, with $\|\rho\|_{L,\nu} \leq \|f\|_{L,\nu}$.*

Proof. The existence of ρ and bound $\|\rho\|_{L,\nu} \leq \|f\|_{L,\nu}$ are just a variation of Lemma 4 where we drop the assumption $f > 0$. The proof is not detailed here since it follows the same lines as the last paragraph of the proof of Lemma 3, decomposing $\boldsymbol{S}_\nu f \boldsymbol{S}_\nu = g^+ - g^-$ with $g^+, g^- \geq 0$ in $\mathcal{K}^1(\mathcal{H})$ such that $|\boldsymbol{S}_\nu f \boldsymbol{S}_\nu| = g^+ + g^-$.

Let us thus prove the uniqueness of ρ. By linearity this amounts to proving that if $\rho \in \mathcal{K}_{L,\nu}(\mathcal{H})$ solves $\rho + \lambda\mathfrak{A}(\rho) = 0$ then $\rho = 0$. Take any such $\rho \in \mathcal{K}_{L,\nu}(\mathcal{H})$; by Lemma 2 we are ensured that $\boldsymbol{L}\rho \boldsymbol{L}^\dagger \in \mathcal{K}^1(\mathcal{H})$. While for $r = 1$ we must still prove uniqueness, for $r \in [0,1)$, according to Lemma 3, there exists a *unique* solution $\rho_r \in \mathcal{K}^1(\mathcal{H})$ to equation (10) where f is replaced by $\tilde{f} = (1-r)\lambda \boldsymbol{L}\rho \boldsymbol{L}^\dagger \in \mathcal{K}^1(\mathcal{H})$. Since ρ satisfies (10) with the same $\tilde{f}$, we must have $\rho_r = \rho$. Moreover we have

$$\mathrm{Tr}\left(|\rho|\right) = \mathrm{Tr}\left(|\rho_r|\right) \leq \mathrm{Tr}\left(|\tilde{f}|\right) = (1-r)\lambda \mathrm{Tr}\left(|\boldsymbol{L}\rho \boldsymbol{L}^\dagger|\right),$$

with $\mathrm{Tr}\left(|\boldsymbol{L}\rho\boldsymbol{L}^\dagger|\right)$ finite and independent of r. By taking the limit for r tending to 1^- we conclude that $\rho = 0$. $\square$

4 Asymptotic Convergence

We now restrict our attention to the space $\mathcal{D}$ of quantum states, i.e. nonnegative operators ρ of trace one. This is justified as by construction, the quantum dynamics (3) preserves the trace and the positivity of ρ_0. We characterize convergence in two ways. First, we prove that every trajectory of (3), in an appropriate Banach space, converges to a unique equilibrium point $\bar{\rho}$ whose support is spanned by the coherent states $\{|\alpha_m\rangle\}_{m=1,2,\ldots,k}$ with $\alpha_m = \alpha\, e^{2i\pi m/k} \in \mathbb{C}$. Second, we identify invariants of the dynamics, which allow to readily determine to which $\bar{\rho}$ any particular initial state ρ_0 would converge.

4.1 Unique limit point

The first step towards proving asymptotic convergence is to identify an efficient Lyapunov function. From Theorem 1 we know that $\|\rho_t\|_{L,\nu}$ is non-increasing for any $\nu \geq 1/2$, but we do not know its decrease rate and to compute its time-derivative in the infinite-dimensional setting we must be careful. The usual solution to this issue is to impose stronger regularity on the solutions than strictly needed, such that sufficient regularity is ensured for the time-derivative.

We take the same approach and restrict our attention to solutions ρ_t of the initial value problem in $\mathcal{K}_{L,\nu}(\mathcal{H})$ with $\nu \geq 3/2$. Such solutions of course also belong to $\mathcal{K}_{L,\nu'}(\mathcal{H})$ for any $\nu' \in [1/2, \nu]$ and hence any associated $\|\rho_t\|_{L,\nu'}$ is non-increasing. We will now characterize the decrease rate for $\nu' = 1/2$, more precisely for

$$V(\rho) = \|\rho_t\|_{L,1/2} - 1\,.$$

Lemma 6. *For any quantum state $\rho \geq 0$ in $\mathcal{K}_{L,\nu}(\mathcal{H})$ with $\nu \geq 3/2$, we have*

$$V(\rho) = \mathit{Tr}\left(\boldsymbol{L}\rho\boldsymbol{L}^\dagger\right) \geq 0 \quad \text{and} \quad V(\mathfrak{L}_{\boldsymbol{L}}(\rho)) \leq -k!\, V(\rho) \leq 0\,.$$

Proof. Define $\phi = \boldsymbol{S}_{3/2}\rho\boldsymbol{S}_{3/2}$ which belongs to $\mathcal{K}^1(\mathcal{H})$ for any $\rho \in \mathcal{K}_{L,\nu}(\mathcal{H})$ with $\nu \geq 3/2$.

For the first statement, noting that a quantum state ρ is positive and of trace one, we get:

$$\begin{aligned}
\mathrm{Tr}\left(\boldsymbol{L}\rho\boldsymbol{L}^\dagger\right) + 1 &= \mathrm{Tr}\left(\left(\boldsymbol{L}\boldsymbol{S}_{3/2}^{-1}\right)\phi\left(\boldsymbol{S}_{3/2}^{-1}\boldsymbol{L}^\dagger\right)\right) + \mathrm{Tr}\left(\boldsymbol{S}_{3/2}^{-1}\phi\boldsymbol{S}_{3/2}^{-1}\right) \\
&= \mathrm{Tr}\left(\frac{\boldsymbol{L}^\dagger\boldsymbol{L}}{(\boldsymbol{I}+\boldsymbol{L}^\dagger\boldsymbol{L})^3}\,\phi + \frac{\boldsymbol{I}}{(\boldsymbol{I}+\boldsymbol{L}^\dagger\boldsymbol{L})^3}\,\phi\right) \\
&= \mathrm{Tr}\left((\boldsymbol{I}+\boldsymbol{L}^\dagger\boldsymbol{L})^{-2}\,\phi\right) = \mathrm{Tr}\left(|(\boldsymbol{I}+\boldsymbol{L}^\dagger\boldsymbol{L})^{-1}\phi(\boldsymbol{I}+\boldsymbol{L}^\dagger\boldsymbol{L})^{-1}|\right) \\
&= \mathrm{Tr}\left(|(\boldsymbol{I}+\boldsymbol{L}^\dagger\boldsymbol{L})^{1/2}\boldsymbol{S}_{3/2}^{-1}\phi\boldsymbol{S}_{3/2}^{-1}(\boldsymbol{I}+\boldsymbol{L}^\dagger\boldsymbol{L})^{1/2}|\right) = \|\rho\|_{L,1/2}\,.
\end{aligned}$$

In the first term of the first line we have a product of a trace-class operator ϕ with two bounded operators, hence we can apply (4); all the other terms are then well-defined.

For the second statement, we check that $\boldsymbol{L}\big(\boldsymbol{L}\boldsymbol{S}_{3/2}^{-1}\phi\boldsymbol{S}_{3/2}^{-1}\boldsymbol{L}^\dagger\big)\boldsymbol{L}^\dagger$ and also $\boldsymbol{L}\big(\boldsymbol{L}^\dagger\boldsymbol{L}\boldsymbol{S}_{3/2}^{-1}\phi\boldsymbol{S}_{3/2}^{-1}\big)\boldsymbol{L}^\dagger$ are products of bounded and trace-class operators, and by applying (4) we are able to get:

$$\begin{aligned}\mathrm{Tr}\left(\boldsymbol{L}\mathfrak{L}_{\boldsymbol{L}}(\rho)\boldsymbol{L}^\dagger\right) &= \mathrm{Tr}\left(\phi\boldsymbol{S}_{3/2}^{-1}\boldsymbol{L}^\dagger[\boldsymbol{L}^\dagger,\boldsymbol{L}]\boldsymbol{L}\boldsymbol{S}_{3/2}^{-1}\right) = -\mathrm{Tr}\left(\phi\boldsymbol{S}_{3/2}^{-1}\boldsymbol{L}^\dagger\boldsymbol{M}\boldsymbol{L}\boldsymbol{S}_{3/2}^{-1}\right)\\ &\leq -k!\,\mathrm{Tr}\left(\phi\boldsymbol{S}_{3/2}^{-1}\boldsymbol{L}^\dagger\boldsymbol{L}\boldsymbol{S}_{3/2}^{-1}\right) = -k!\,V(\rho)\,.\end{aligned}$$

Here we have used $\boldsymbol{M}$ from (5) and the rough estimate $\boldsymbol{M} \geq (\boldsymbol{N}+\boldsymbol{I})(\boldsymbol{N}+2\boldsymbol{I})...(\boldsymbol{N}+k\boldsymbol{I}) - (\boldsymbol{N}+\boldsymbol{I})\boldsymbol{N}...\boldsymbol{N} \geq (\boldsymbol{N}+\boldsymbol{I})k! \geq k!\boldsymbol{I}$, together with positivity properties for the inequality. □

This means that $V(\rho)$ is an exponential Lyapunov function. We finally apply the infinite-dimensional version of the LaSalle invariance principle to $V(\rho)$.

Theorem 2. *Consider the unique trajectory $[0,+\infty[\ni t\mapsto \rho(t)\in\mathcal{K}_{L,\nu}(\mathcal{H})$ solution of (3), with $\nu\geq 3/2$. Then there exists $\bar\rho\in\mathcal{K}_{L,\nu}(\mathcal{H})$, with support in*

$$\mathcal{H}_{\alpha,k} = span\Big\{|\alpha_m\rangle : \alpha_m = \alpha\, e^{2i\pi m/k}\ ,\ \ m=1,2,...,k\Big\},$$

such that ρ converges to $\bar\rho$ for $\|\cdot\|_{L,\nu'}$-norm with $\nu'<\nu$.
Moreover, we have exponential convergence towards $\mathcal{H}_{\alpha,k}$ in the sense:

$$Tr\left(\left|\boldsymbol{L}(\rho(t)-\bar\rho)\boldsymbol{L}^\dagger\right|\right) \leq Tr\left(\boldsymbol{L}|\rho_0-\bar\rho|\boldsymbol{L}^\dagger\right)\ e^{-k!\,t}\,.$$

Proof. For any given $\rho_0\in\mathcal{K}_{L,\nu}(\mathcal{H})$, from Section 3 we know that $\rho(t)\in\mathcal{K}_{L,\nu}(\mathcal{H})$ for all $t\geq 0$ and according to Lemma 6 the function $f(t)=V(\rho(t))$ converges exponentially to zero as t increases. For any $\nu'>\nu$, the injection of $\mathcal{K}_{L,\nu'}(\mathcal{H})$ into $\mathcal{K}_{L,\nu}(\mathcal{H})$ is compact. Thus the trajectory $\{\rho(t)\mid t\geq 0\}$ is precompact in $\mathcal{K}_{L,\nu}(\mathcal{H})$, which means that it must have an adherent point $\bar\rho\in\mathcal{K}_{L,\nu}(\mathcal{H})$ for t tending towards infinity.

Lemma 6 implies $\mathrm{Tr}\left(\boldsymbol{L}\bar\rho\boldsymbol{L}^\dagger\right)=0$, i.e. $\bar\rho$ is a steady state and its support is contained in the kernel of $\boldsymbol{L}$. The latter is spanned by vectors satisfying $\boldsymbol{a}^k|\psi\rangle=\alpha^k|\psi\rangle$ and writing $|\psi\rangle=\sum_{m\in\mathbb{N}}\psi_m|m\rangle$ we get the recurrence relation

$$\psi_{m+k} = \frac{\alpha^k}{\sqrt{(m+k)...(m+2)(m+1)}}\,\psi_m \quad \text{for all } m\geq 0.$$

This leaves k degrees of freedom to initialize the recurrence(s), which besides that is satisfied by the k states mentioned in the statement as span of $\mathcal{H}_{\alpha,k}$. Thus the kernel of $\boldsymbol{L}$ indeed coincides with $\mathcal{H}_{\alpha,k}$.

For the first part, there remains to show that the adherence point is unique. It is well-known, see e.g. [14, Chapter 9], that the semigroup associated to the Lindblad master equation is a (non-strict) contraction for the trace distance. Hence the distance between our trajectory $\rho_t \in \mathcal{K}_{L,\nu}(\mathcal{H})$ and the particular solution $\rho = \bar{\rho}$ of (3) is non-increasing, i.e. $t \mapsto \mathrm{Tr}\left(|\rho(t) - \bar{\rho}|\right)$ is non-increasing. Consequently the adherent point $\bar{\rho}$ is unique and $\rho(t)$ converges towards $\bar{\rho}$ in $\mathcal{K}_{L,\nu}(\mathcal{H})$.

For exponential convergence, decompose $\rho(0) - \bar{\rho} = \zeta_0^+ - \zeta_0^-$ (positive and negative part) where the evolution of ζ_t^+ and ζ_t^- are defined by

$$\frac{d}{dt}\zeta^+ = \mathfrak{L}(\zeta^+), \quad \zeta^+(0) = \zeta_0^+$$
$$\frac{d}{dt}\zeta^- = \mathfrak{L}(\zeta^-), \quad \zeta^-(0) = \zeta_0^-$$

therefore, by linearity, $\rho(t) - \bar{\rho} = \zeta_t^+ - \zeta_t^-$ and we get

$$\begin{aligned} \mathrm{Tr}\left(|\boldsymbol{L}(\zeta_t^+ - \zeta_t^-)\boldsymbol{L}^\dagger|\right) &\leq \mathrm{Tr}\left(|\boldsymbol{L}\zeta_t^+\boldsymbol{L}^\dagger|\right) + \mathrm{Tr}\left(|\boldsymbol{L}\zeta_t^-\boldsymbol{L}^\dagger|\right) \\ &= \mathrm{Tr}\left(\boldsymbol{L}(\zeta_t^+ + \zeta_t^-)\boldsymbol{L}^\dagger\right) \leq \mathrm{Tr}\left(\boldsymbol{L}|\rho_0 - \bar{\rho}|\boldsymbol{L}^\dagger\right)\, e^{-k!\,t}\,, \end{aligned}$$

where the last inequality follows from Lemma 6 and $\zeta_0^+ + \zeta_0^- = |\rho(0) - \bar{\rho}|$. □

Regarding exponential convergence, since the kernel of $\boldsymbol{L}$ coincides with $\mathcal{H}_{\alpha,k}$ and the spectrum of $\sqrt{\boldsymbol{L}^\dagger \boldsymbol{L}}$ has no accumulation point at 0, the role of the $\boldsymbol{L}$ operators is essentially to project $\rho(t)$ onto the complement of $\mathcal{H}_{\alpha,k}$. I.e. if $\mathcal{P}_k$ is the orthonormal projector from $\mathcal{H}$ onto $\mathcal{H}_{\alpha,k}$, then there exists some $c > 0$ and $c' > 0$ (depending on ρ_0) such that

$$\mathrm{Tr}\left(|(\boldsymbol{I} - \mathcal{P}_k)(\rho(t) - \bar{\rho})(\boldsymbol{I} - \mathcal{P}_k)|\right) \leq\; c\, \mathrm{Tr}\left(\boldsymbol{L}|\rho_0 - \bar{\rho}|\boldsymbol{L}^\dagger\right)\, e^{-k!\,t} \;\leq\; c'\, e^{-k!\,t}\,.$$

This expresses exponential convergence towards $\mathcal{H}_{\alpha,k}$ in the trace-norm. The inclusion of the unbounded operator $\boldsymbol{L}$ on the left-hand side makes the statement of Theorem 2 slightly stronger.

Note that the Hermitian operators with support on $\mathcal{H}_{\alpha,k}$ belong to $\mathcal{K}_{L,\nu}(\mathcal{H})$ for all $\nu \geq 0$. They form a k^2-dimensional real subspace of $\mathcal{K}_{L,\nu}(\mathcal{H})$, since $\mathcal{H}_{\alpha,k}$ is a k-dimensional complex subspace of $\mathcal{H}$. Thus the exponential convergence neglects k^2 directions in $\mathcal{K}_{L,\nu}(\mathcal{H})$. The behavior of the system in these directions, i.e. how the limit $\bar{\rho}$ depends on ρ_0, is clarified in the following section.

4.2 Invariants of the dynamics

We will derive invariants of the dynamics in terms of bounded Hermitian operators on $\mathcal{H}$, i.e., in terms of physical observables. For this, we first define a continuous extension to any $\rho_0 \in \mathcal{K}^1(\mathcal{H})$ of the superoperator mapping ρ_0 to its corresponding limit $\bar{\rho}$ according to Theorem 2.

Consider the map $\mathfrak{K}_\infty$ from $\mathcal{K}^f(\mathcal{H})$ to $\mathcal{K}^1(\mathcal{H})$ defined as follows: for any $\xi \in \mathcal{K}^f(\mathcal{H})$, $\mathfrak{K}_\infty(\xi) = \lim_{t\mapsto+\infty} \rho(t)$ where $\rho(t)$ is the unique solution of (9) with $\rho_0 = \xi$. By construction,

$\mathfrak{K}_\infty$ is linear, trace preserving, completely positive and a contraction for the nuclear norm [14, Chapter 9]:

$$\forall \xi_1, \xi_2 \in \mathcal{K}^f(\mathcal{H}), \quad \mathrm{Tr}\left(\left|\mathfrak{K}_\infty(\xi_1) - \mathfrak{K}_\infty(\xi_2)\right|\right) \leq \mathrm{Tr}\left(|\xi_1 - \xi_2|\right).$$

Since $\mathcal{K}^f(\mathcal{H})$ is dense in $\mathcal{K}^1(\mathcal{H})$, we can extend the domain of definition of $\mathfrak{K}_\infty$ to all $\xi \in \mathcal{K}^1(\mathcal{H})$ by continuity. From there, we get the following result.

Theorem 3. *There exist k^2 linearly independent, bounded Hermitian operators $\boldsymbol{Q}_{m,m'} \in \mathcal{K}^\infty(\mathcal{H})$, $m, m' = 1, 2, ..., k$, which are invariant under the dynamics (3), i.e. for which*

$$Tr\left(\boldsymbol{Q}_{m,m'}\,\rho_t\right) = Tr\left(\boldsymbol{Q}_{m,m'}\,\rho_0\right)$$

for any trajectory $[0, +\infty) \ni t \mapsto \rho_t \in \mathcal{K}_{L,\nu}(\mathcal{H})$ with $\nu \geq 3/2$.

Moreover, the linear space of invariant Hermitian operators spanned by $\{\boldsymbol{Q}_{m,m'}\}_{m,m'=1...k}$ contains in particular the k operators

$$\boldsymbol{Q}_m^{\cos} = \sum_{n \in \mathbb{N}} \cos\left(\frac{2\pi m n}{k}\right) |n\rangle\langle n| \qquad \textit{for} \quad m = 0, 1, ..., \lceil \tfrac{k-1}{2} \rceil\,;$$

$$\boldsymbol{Q}_m^{\sin} = \sum_{n \in \mathbb{N}} \sin\left(\frac{2\pi m n}{k}\right) |n\rangle\langle n| \qquad \textit{for} \quad m = 1, ..., \lfloor \tfrac{k-1}{2} \rfloor\,.$$

Proof. The image of $\mathfrak{K}_\infty$ has support on $\mathcal{H}_{\alpha,k}$, hence we can write

$$\begin{aligned}\mathfrak{K}_\infty(\rho_0) \;=\; & \sum_{m=1}^{k}\sum_{m'=1}^{m-1} Q_{m,m'}(\rho_0)\left(|\alpha_m\rangle\langle\alpha_{m'}| + |\alpha_{m'}\rangle\langle\alpha_m|\right) \\ & + \sum_{m=1}^{k} Q_{m,m}(\rho_0)\,|\alpha_m\rangle\langle\alpha_m| \\ & + \sum_{m=1}^{k}\sum_{m'=m+1}^{k} Q_{m,m'}(\rho_0)\left(i|\alpha_m\rangle\langle\alpha_{m'}| - i|\alpha_{m'}\rangle\langle\alpha_m|\right)\end{aligned}$$

with $Q_{m,m'}(\rho_0)$ real. Since $\mathfrak{K}_\infty$ is a linear continuous map from $\mathcal{K}^1(\mathcal{H})$ to $\mathcal{K}^1(\mathcal{H})$, the $Q_{m,m'} : \mathcal{K}^1(\mathcal{H}) \ni \rho_0 \mapsto Q_{m,m'}(\rho_0)$ are linear continuous maps from $\mathcal{K}^1(\mathcal{H})$ to $\mathbb{R}$. Thus they belong to the dual of $\mathcal{K}^1(\mathcal{H})$ corresponding to $\mathcal{K}^\infty(\mathcal{H})$, the set of bounded operators on $\mathcal{H}$:

$$Q_{m,m'}(\rho) \;=\; \mathrm{Tr}\left(\boldsymbol{Q}_{m,m'}\,\rho\right)$$

where $\boldsymbol{Q}_{m,m'}$ are *bounded* Hermitian operators. Since any Hermitian operator $\bar{\rho}$ with support on $\mathcal{H}_{\alpha,k}$ belongs to the image of $\mathfrak{K}_\infty$, and this is a k^2-dimensional set, the k^2 operators $\{\boldsymbol{Q}_{m,m'}\}$ must indeed be linearly independent. Moreover, since all ρ_t along the trajectory defined by (3) starting at $\rho_0 \in \mathcal{K}_{L,\nu}(\mathcal{H})$, $\nu \geq 3/2$, are associated to the same $\bar{\rho} = \mathfrak{K}_\infty(\rho_t)$, we have that $\mathrm{Tr}\left(\boldsymbol{Q}_{m,m'}\,\rho_t\right)$ is constant along such trajectories.

The particular operators $\boldsymbol{Q}_m^{\cos}$ and $\boldsymbol{Q}_m^{\sin}$ are indeed bounded and e.g.

$$\mathrm{Tr}\,(\boldsymbol{Q}_m^{\cos}\xi) = \sum_{n\in\mathbb{N}} \cos\left(\frac{2\pi mn}{k}\right)\,\xi_n$$

for any $\xi \in \mathcal{K}^1(\mathcal{H})$, where we have written $\xi_n = \langle n|\xi|n\rangle$; let also $\xi_{\ell,n} = \langle \ell|\xi|n\rangle$. Computing $\mathfrak{L}_{\boldsymbol{L}}(\rho)$ explicitly in the canonical basis, we get

$$\begin{aligned}
&\textstyle\sum_{n\in\mathbb{N}} \cos\left(\frac{2\pi mn}{k}\right)\,\frac{d}{dt}\xi_n \\
&= \sum_{n\in\mathbb{N}} \xi_{n+k}\,(n+k)...(n+1)\,\left(\cos\left(\tfrac{2\pi mn}{k}\right) - \cos\left(\tfrac{2\pi m(n+k)}{k}\right)\right) \\
&\quad -\frac{\alpha^k}{2}\sum_{n\in\mathbb{N}} (\xi_{n,n+k}+\xi_{n+k,n})\,\sqrt{(n+k)...(n+1)}\,\left(\cos\left(\tfrac{2\pi mn}{k}\right) - \cos\left(\tfrac{2\pi m(n+k)}{k}\right)\right) \;= 0\,,
\end{aligned}$$

which indeed proves invariance of $\boldsymbol{Q}_m^{\cos}$ along trajectories; for $\boldsymbol{Q}_m^{\sin}$ the proof is exactly the same. □

Note that the pure states on $\mathcal{H}_{\alpha,k}$ span a space of dimension k, thus in this sense having k invariants can provide significant insight about the link between ρ_0 and its associated $\bar{\rho}$. In the present case, this link is particularly meaningful.

We have $\boldsymbol{Q}_0^{\cos} = \boldsymbol{I}$, so this particular invariant just expresses conservation of the trace of ρ. The other particular invariants feature as eigenstates so-called "Schrödinger cat states" of the harmonic oscillator, more soberly called coherent quantum superpositions of mesoscopic states, and whose general form is defined in equation (2). More precisely, denoting

$$|C_\alpha^\ell\rangle = \frac{1}{\vartheta}\sum_{m=1}^{k} e^{2i\pi\ell m/k}\,|\alpha_m\rangle$$

with ϑ a normalizing constant, a few computations based on the definitions directly yield:

$$\boldsymbol{Q}_m^{\cos}\,|C_\alpha^\ell\rangle \;= \cos\left(\tfrac{2\pi\ell m}{k}\right)\,|C_\alpha^\ell\rangle \quad\text{and}\quad \boldsymbol{Q}_m^{\sin}\,|C_\alpha^\ell\rangle \;= \sin\left(\tfrac{2\pi\ell m}{k}\right)\,|C_\alpha^\ell\rangle\;.$$

The Schrödinger cats $|C_\alpha^\ell\rangle$ are specifically quantum states with no classical analogue, and they are a promising tool towards implementing quantum IT applications [12], thanks to their inherent insensitivity to part of the typical perturbations present in quantum systems. The known invariants allow us to predict towards which fraction of each cat $|C_\alpha^\ell\rangle$ an arbitrary initial state ρ_0 will evolve. This can be useful for investigating more precisely the sensitivity of information encoded in such cat states to typical perturbations.

The authors of [12] discuss the case $k=2$ in this direction. In that case, the k particular invariants of Theorem 3 reduce to the identity and the parity operator $\sum_{n\in\mathbb{N}}(-1)^n\,|n\rangle\langle n|$. Explicit expressions for the $k^2-k=2$ remaining linearly independent invariants are also provided in terms of Bessel functions. Generalizing these expressions for $k>2$ remains for future work.

5 Conclusion

We have proved, with a rigorous infinite-dimensional treatment, that a harmonic oscillator governed by a Lindblad master equation where the typical drive and loss channels are k-photon processes instead of single-photon ones, converges to a protected subspace spanned by k coherent states of the same amplitude and uniformly distributed phases. We have also proved the existence of k^2 invariant bounded observables (Hermitian operators), i.e. whose expectation value is conserved by the dynamics. Knowing these invariants would allow to directly predict the final state $\bar{\rho}$ towards which a given ρ_0 converges. We have provided explicit expressions for k such invariant observables, whose eigenstates are Schrödinger cat states belonging to the protected subspace and which appear as robust candidates to encode quantum information.

The infinite-dimensional arguments use the Hille-Yosida theorem and a Lyapunov analysis in a particular family of Banach spaces $\mathcal{K}_{L,\nu}(\mathcal{H})$ whose metric is built directly from the k-photon Lindblad operator. For practical purposes, our contribution is to show that there exists a self-consistent way to indeed have infinite-dimensional convergence to a so-called protected subspace in this model, and to clarify convergence speed with various metrics. We may not have identified the largest functional spaces (respective values of ν in $\mathcal{K}_{L,\nu}(\mathcal{H})$) for this convergence.

We guess that the methods used here to study the well-posedness of our particular infinite-dimensional Lindbald master equation can be adapted to other models appearing in reservoir engineering for cavity and/or circuit quantum electrodynamics.

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Appendix

Theorem 4 (Hille-Yosida in Banach spaces [4, Chap.7]). *Let E be a Banach space and A an m-accretive operator on E, i.e., the domain $D(A)$ of A is dense in E and for every $\lambda > 0$, $I + \lambda A$ is a bijection from $D(A)$ into E with, for all $u \in E$, $\|(I + \lambda A)^{-1} u\| \leq \|u\|$. Then for any $u_0 \in D(A)$, there exists a unique function $[0, +\infty[\ni t \mapsto u(t) \in D(A)$ that is continuously differentiable such that*

$$\frac{d}{dt} u + A(u) = 0 \text{ for } t \in [0, +\infty[, \quad u(0) = u_0.$$

Moreover $\forall t \geq 0$, $\|u(t)\| \leq \|u_0\|$ and $\|\frac{d}{dt} u(t)\| = \|A(u(t))\| \leq \|A(u_0)\|$